

INTRODUCTION TO MODEL THEORY OF VALUED FIELDS

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ABSTRACT. These are lecture notes from a graduate course on p -adic and motivic integration (given at BGU). The main topics are: Quantifier elimination in the p -adics, rationality of p -adic zeta functions and their motivic analogues, basic model theory of algebraically closed valued fields, motivic integration following Hrushovski and Kazhdan, application to the Milnor fibration. Background: basic model theory and a bit of algebraic geometry

1. p -ADIC INTEGRATION

1.1. **Counting solutions in finite rings.** Consider a system \mathbf{X} of polynomial equations in n variables over \mathbb{Z} . Understanding the set $\mathbf{X}(\mathbb{Z})$ is, generally, very difficult, so we make (rough) estimates: Given a prime number $p > 0$, we consider the set $\mathbf{X}(\mathbb{Z}/p^k\mathbb{Z})$ of solutions of \mathbf{X} in the finite ring $\mathbb{Z}/p^k\mathbb{Z}$, viewing them as finer and finer approximations, as k increases. Each such set is finite, and we may hope to gain insight on the set of solutions by understanding the behaviour of the sequence $a_k = \#\mathbf{X}(\mathbb{Z}/p^k\mathbb{Z})$, which we organise into a formal power series $P_{\mathbf{X}}(T) = \sum_{k=0}^{\infty} a_k T^k$, called the *Poincaré series* for \mathbf{X} . Igusa proved:

Poincaré series

Theorem 1.1.1. *The series $P_{\mathbf{X}}(T)$ is a rational function of T*

Example 1.1.2. Assume that \mathbf{X} is the equation $x_1 = 0$. There are then $a_k = p^{(n-1)k}$ solutions in $\mathbb{Z}/p^k\mathbb{Z}$, and $P_{\mathbf{X}}(T) = \sum (p^{n-1}T)^k = \frac{1}{1-p^{n-1}T}$ \square

To outline the proof, we first note that the contribution of a given integer x to the sequence (a_i) is determined by the sequence (x_k) of its residues mod p^k . Each such sequence has the property that $\pi_{k,l}(x_k) = x_l$, where $\pi_{k,l} : \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^l\mathbb{Z}$ for $k \geq l$ is the residue map. The set of sequences (x_k) with this property forms a ring (with pointwise operations), called the ring of *p -adic integers*, \mathbb{Z}_p . If the first k entries of such a sequence are 0, one may view this sequence as “close to 0, up to the k -th approximation”. This notion of closeness can be formalised by defining the *absolute value* $|x|$ of an element $x = (x_i)$ to be p^{-k} , where k is the smallest i for which $x_i \neq 0$ (and $|0| = 0$). The number k is called the *valuation* $v_p(x)$ of x . Thus, x is k -close to 0 if $|x| \leq p^{-k}$ (one could choose a different base instead of p to obtain the same topology; the motivation for choosing p will soon become apparent.) It is easy to check that \mathbb{Z}_p is a local ring whose maximal ideal is the set of

p -adic integers

absolute value

valuation

elements of norm smaller than 1. As a topological space it is *complete* (every Cauchy sequence converges) and compact.

We now assume, for simplicity, that \mathbf{X} is given by a single equation, $F(\bar{x}) = 0$. Thus,

$$\begin{aligned} \alpha_k &= \#(\{\bar{x} \in \mathbb{Z}_p^n : F(\pi_k(\bar{x})) = 0\}/B_k) = \#(\{\bar{x} \in \mathbb{Z}_p^n : \pi_k(F(\bar{x})) = 0\}/B_k) = \\ &= \#(\{\bar{x} \in \mathbb{Z}_p^n : |F(\bar{x})| \leq p^{-k}\}/B_k) \end{aligned} \quad (1)$$

where B_k is the (n -dimensional) ball $|x_i| \leq p^{-k}$ of radius p^{-k} . The set $\mathbf{X}_k = \{\bar{x} \in \mathbb{Z}_p^n : |F(\bar{x})| \leq p^{-k}\}$ can be viewed as a neighbourhood of the set of solutions $\mathbf{X}(\mathbb{Z}_p)$, and we are counting the number of balls in this neighbourhood. If we were given a measure μ on \mathbb{Z}_p^n which is invariant under translations (and for which \mathbf{X}_k and B_k are measurable), we could thus write $\alpha_k = \frac{\mu(\mathbf{X}_k)}{\mu(B_k)}$. In fact, we have such a measure: It is the *Haar measure*, the unique (after normalisation) translation invariant measure that exists for any locally compact topological group. Since our group is actually compact, we may normalise so that the whole group $B_0 = \mathbb{Z}_p^n$ has measure 1. We may then compute the measure of the ball B_k : there are p^{nk} disjoint translates of it that cover B_0 , hence $\mu(B_k) = p^{-nk}$.

The existence of a measure allows us to integrate, and the above calculations suggest that our series $P_{\mathbf{X}}$ is related to integrating $|F(x)|$. In fact, we have

$$\begin{aligned} \int_{B_0} |F(x)|^s d\mu &= \sum_{k=0}^{\infty} \mu(\{x \in B_0 : |F(x)| = p^{-k}\}) p^{-ks} = \\ &= \sum_{k \geq 0} p^{-ks} (\mu(\mathbf{X}_k) - \mu(\mathbf{X}_{k+1})) = \\ &= \sum_{k \geq 0} \alpha_k p^{k(-n-s)} - \alpha_{k+1} p^{-n} p^{k(-n-s)} = \\ &= 1 + \sum_{k \geq 1} \alpha_k (p^{k(-n-s)} - p^{-n} p^{(k-1)(-n-s)}) = \\ &= 1 + \sum_{k \geq 1} \alpha_k (1 - p^s) p^{k(-n-s)} = 1 + (1 - p^s) \sum_{k \geq 1} \alpha_k p^{k(-n-s)} \end{aligned} \quad (2)$$

This (\mathbb{C} -valued) function, denoted $Z_F(s)$, is called the *Igusa zeta function* associated to F . We thus have that

$$Z_F(s) = p^s + (1 - p^s) P_{\mathbf{X}}(p^{-n-s}) \quad (3)$$

In particular, proving Theorem 1.1.1 amounts to proving that $Z_F(s)$ is a rational function of p^s .

What did we gain from this translation? We have seen in Example 1.1.2 that Z_F is rational when F is a coordinate function. It turns out that around

Haar measure

Igusa zeta function

a *smooth* point of \mathbf{X} , there is an (appropriately defined) analytic change of coordinates mapping a (p -adic) neighbourhood to a coordinate plane, and F to a coordinate function (this is similar to the complex situation). There is, further, a change of coordinates formula for integration, using which one may reduce to a case similar to the example. When \mathbf{X} is not smooth, one must first use resolution of singularities, and integrate there (using, again, the change of variable formula to relate the two integrals).

1.1.3. *More general integrals.* An element of $\mathbf{X}(\mathbb{Z}/p^k\mathbb{Z})$ need not come from a solution in \mathbb{Z}_p . If we are interested only in those solutions that do lift to solutions in \mathbb{Z}_p , we are led to consider the *Serre series* $\sum b_k T^k$ of \mathbf{X} , where $b_k = \#(\mathbf{X}(\mathbb{Z}_p)/B_k)$. We note that $\mathbf{X}(\mathbb{Z}_p)$ is not invariant under translation by B_k , so to express the coefficients in terms of measure, we rewrite

Serre series

$$b_k = \#((\mathbf{X}(\mathbb{Z}_p) + B_k)/B_k) = \mu(\tilde{\mathbf{X}}_k)/\mu(B_k) \quad (4)$$

where

$$\tilde{\mathbf{X}}_k = \{\bar{x} \in \mathbb{Z}_p^n : \exists \bar{y} \in \mathbf{X}(\mathbb{Z}_p) (|\bar{x} - \bar{y}| \leq p^{-k})\} \quad (5)$$

is the set of points that are of distance at most p^{-k} from solutions. We thus see that the coefficients can again be written as integrals, but the domain is not given by polynomial equations, but rather by conditions that involve *quantifiers*: they are first order formulas in the language of *p -adic fields*, i.e., whose basic relations involve, on top of the field operations, the valuation function.

What can be said about such integrals? Denef proved the following generalisation of Theorem 1.1.1

Theorem 1.1.4. *Let $n_i = \int_{X_i} p^{\alpha(x,i)} d\mu$, where X_i are uniformly definable subsets of \mathbb{Z}_p^n , and α is a definable integer valued function. Then $\sum_k n_k T^k$ is a rational function*

The main ingredient in the proof of Theorem 1.1.4 is an analysis of the shape of definable subsets the p -adic numbers: it is a quantifier elimination result (in a suitable language), due to Ax–Kochen and Macintyre. We now explain the statement and the proof of this result in more detail.

1.2. **Quantifier elimination in the p -adics.** For extra concreteness, we now outline some structure of \mathbb{Z}_p , and the proof of quantifier elimination in the p -adics. The original proof is from [3, 4] and [17]. See also [8, Ch. 4].

It will now be more convenient notationally to work with the valuation $v = v_p$. It follows from the existence of the valuation that \mathbb{Z}_p is an integral domain. Its field of fractions, \mathbb{Q}_p , is the field of *p -adic numbers*. The valuation and absolute value extend from \mathbb{Z}_p to \mathbb{Q}_p by multiplicativity, and take values, respectively, in \mathbb{Z} and in $\mathbb{R}_{\geq 0}$. We note that \mathbb{Z}_p is the set of elements with non-negative valuation. The pair (\mathbb{Q}_p, v) is an example of a valued field:

 p -adic numbers

Definition 1.2.1. A *valued field* is a triple (K, Γ, v) , where K is a field, Γ is an ordered abelian group, and $v : K^\times \rightarrow \Gamma$ such that for all $x, y \in K^\times$

- (1) $v(xy) = v(x) + v(y)$
- (2) $v(x + y) \geq \min(v(x), v(y))$ if $x + y \neq 0$

Since Γ is now an arbitrary group, it no longer makes sense to consider p^γ for $\gamma \in \Gamma$. However, we will still wish to use the absolute value notation (mostly for geometric intuition), and we do so by formally inverting the order on Γ , and using multiplicative notation for the group operation. In this case, we denote v by $|\cdot|$, as before. The second condition then becomes $|x + y| \leq \max(|x|, |y|)$, and is called the *ultrametric inequality*.

We add an element 0 (or ∞ when using the additive notation) to Γ , and extend the structure by specifying $|0| = 0$ (and $v(0) = \infty$). With the obvious conventions, the conditions in Definition 1.2.1 are now valid for all $x, y \in K$.

Exercise 1.2.2. Show that $(\mathbb{Q}_p, \mathbb{Z}, v_p)$ is a valued field

As in \mathbb{Q}_p , the subset \mathcal{O}_v consisting of elements with non-negative valuation is a local sub-ring of K , called the *valuation ring*. Its fraction field is K , and its maximal ideal is the set \mathcal{M}_v of elements with positive valuation. The *residue field* \bar{K} is $\mathcal{O}_v/\mathcal{M}_v$.

Exercise 1.2.3. Let (K, Γ, v) be a valued field

- (1) Show that if $a \in K$ is a root of unity, then $v(a) = 0$
- (2) Show that if K and \bar{K} do not have the same characteristic, then K has characteristic 0
- (3) Let $\gamma \in \Gamma$. Show that the relation $v(x - y) \geq \gamma$ is an equivalence relation on K
- (4) Show that if $v(x) > v(y)$, then $v(x + y) = v(y)$ (geometrically, “every triangle is isosceles”)
- (5) Assume that $p(x)$ is a monic polynomial over \mathcal{O}_v , that has a root $a \in K$. Then $a \in \mathcal{O}_v$ (i.e., \mathcal{O}_v is integrally closed)

We view a valued field as a structure for the two-sorted language with a sort \mathbf{VF} for the field, and another sort $\mathbf{\Gamma}$ for the value group. The language includes (initially) the ring language on \mathbf{VF} , the ordered group language on $\mathbf{\Gamma}$, and a function symbol $v : \mathbf{VF} \rightarrow \mathbf{\Gamma}$ for the valuation. It is clear that the axiom of valued fields are expressible in this language.

What is the theory of \mathbb{Q}_p in this language? We first note that the residue field is completely determined: it is \mathbb{F}_p , and we have representative for all the classes in \mathbb{Q}_p . The value group is \mathbb{Z} , so a *discretely* ordered abelian group (has a minimal positive element). It will turn out that these are essentially the only axioms, but currently, we give the following definition.

Definition 1.2.4. A *p-adic field* is a valued field (K, Γ, v) , such that Γ has a minimal positive element 1 , $v(p) = 1$, and the residue field is \mathbb{F}_p .

It is clear that this is an elementary class, and that \mathbb{Q}_p is p-adic. To obtain quantifier elimination, we will need to understand which equations can be

solved in p -adic fields. We will need, in particular, to have quantifier elimination in the value group, so we start with it.

1.2.5. *Warm up: quantifier elimination in \mathbb{Z} -groups.* The theory of \mathbb{Z} in the language of ordered groups does not eliminate quantifiers: The set of even elements in \mathbb{Z} is definable by $\exists y(y + y = x)$, but not without quantifiers. More generally, the set of elements divisible by n , for natural n , is definable, but not quantifier free. To remedy this, we introduce relations D_n , interpreted in \mathbb{Z} as $n\mathbb{Z}$.

Proposition 1.2.6. *The theory of $(\mathbb{Z}, +, 0, 1, <, (n\mathbb{Z})_{n \in \mathbb{N}})$ in the language for ordered groups, expanded by the divisibility predicates D_n , eliminates quantifiers.*

For a theory \mathcal{T} , by a \mathcal{T} -structure we mean a sub-structure of a model of \mathcal{T} . Let $\tau : A \rightarrow M$ a homomorphism from a \mathcal{T} -structure to a model M . If $\phi(\bar{a}, \bar{x})$ is a formula with $\bar{a} \in A$, we denote by ϕ^τ the formula $\phi(\tau(\bar{a}), \bar{x})$. In the proof of Proposition 1.2.6, we will utilise the following criterion (see also [18] or [21]):

Proposition 1.2.7. *A theory \mathcal{T} eliminates quantifiers if for any \mathcal{T} -structure A , any models M_1 and M_2 containing it, and any quantifier free formula ϕ over A , if ϕ is satisfiable in M_1 , then it is satisfiable in M_2 .*

Proof. If not, there is (by induction on the complexity of the formula) a formula $\psi(\bar{x}) = \exists y(\phi(\bar{x}, y))$ (with ϕ quantifier free), that is not equivalent to any quantifier-free formula. Let Σ be the set of quantifier-free formulas implied by ψ . Since ψ is not quantifier-free, we may by compactness find a model M_2 and a tuple \bar{a} satisfying Σ but not ψ .

Let p be the quantifier-free type of \bar{a} . We note that any tuple satisfying p is isomorphic to \bar{a} as a structure. Again, since ψ is not quantifier-free, it is consistent with p (otherwise, ψ and some formula θ from p are contradictory, so the negation of θ is in Σ , contradicting that \bar{a} satisfies both Σ and p). Thus, we may realise (a copy of) \bar{a} in a model M_1 , where $\psi(\bar{a})$ holds. So $\phi(\bar{a}, y)$ is satisfiable in M_1 but not in M_2 . \square

Remark 1.2.8. We note that the criterion in Prop. 1.2.7 becomes “harder” if we increase A . There are at least two ways to use this observation. First, if A is contained in a structure A_1 such that every embedding of A in a model extends to an embedding of A_1 , we may assume that $A = A_1$ to start with.

On the other hand, assume $b \in M_1$. To extend the situation from A to the substructure $A(b)$ generated by b , we need to find $b' \in M_2$ satisfying the same quantifier-free type over A as b . This appears harder than the criterion, but it becomes equivalent if we assume that M_2 is a *saturated model*, i.e., any quantifier-free type over a set of smaller cardinality than the cardinality of M_2 is satisfied in M_2 . This assumption is harmless since (assuming GCH) any model has a saturated elementary extension. \square

saturated model

Let \mathcal{T} be the theory of \mathbb{Z} as an ordered abelian group, as in Prop. 1.2.6. We note that given for any non-zero integer n , there is, in any model M of

\mathcal{T} , a well defined map $a \mapsto \frac{a}{n}$, assigning to a the smallest element b of M such that $nb \geq a$.

Proof of Prop. 1.2.6. We use Prop. 1.2.7. Let A be a \mathcal{T} -structure, i.e., A is an ordered abelian group. We first claim that we may assume that A is closed under division by integers: given n , there is, for any $a \in A$, a unique $0 \leq i < n$ for which $a + i$ is divisible by n (since this is a first order statement true in \mathbb{Z}). This is expressed as the quantifier free relation $D_n(a + i)$, hence depends only on the structure A (and not on the embedding in a model). It follows that we may assume that A is closed under division inside M_1 and M_2 (in other words, A is a *pure subgroup* of the M_i : M_i/A is torsion-free).

pure subgroup

We need to consider conjunctions of formulas of the following forms (and their negations):

- (1) $nx = a$
- (2) $nx < a$
- (3) $D_m(nx + a)$

Where m, n are non-zero integers, and $a \in A$. We first note that if the conjunction contains a formula of the first kind, then it has a solution in the model if and only if $D_n(a)$ holds. In particular, it does not depend on the model. Furthermore, the solution is unique, and is in A , so the validity of the rest of the formula is determined.

A formula of the second kind is equivalent to $x < \frac{a}{n}$, so including the inequations, a formula is equivalent to a finite disjunction of formulas $a < x < b \wedge \theta(x)$ with $a, b \in A \cup \{\infty, -\infty\}$, and θ a boolean combination of formulas of the third kind. It suffices to deal with each disjunct separately.

As mentioned above, the residue of an element $a \in A$ modulo an integer m is a well defined element of $\mathbb{Z}/m\mathbb{Z}$. Hence, satisfiability of such a formula is completely determined by the theory of \mathbb{Z} (essentially via the Chinese remainder theorem). In particular, it does not depend on the model. \square

Remark 1.2.9. It is easy to extract from the proof the precise set of axioms used. Hence we have an explicit description of the theory of \mathbb{Z} \square

Exercise 1.2.10. Let M be a model of \mathcal{T} . Show that $M = \mathbb{Z} \times H$, where H is a uniquely divisible ordered abelian group, and the order on M is lexicographic.

1.2.11. *The theory of the p -adics.* To show quantifier-elimination for \mathbb{Q}_p , we will show a stronger result: we will describe explicitly a theory which admits quantifier-elimination (in a suitable language), and which is satisfied by \mathbb{Q}_p .

As we have seen, to have quantifier-elimination, we need to be able to determine the existence of solutions to quantifier-free formulas in one variable. In (valued) fields, a central example is a polynomial equation in one variable. If $p(x)$ is a monic polynomial over \mathcal{O}_v , and $p(a) = 0$ for some a (necessarily in \mathcal{O}_v , by Exercise 1.2.3), then $\bar{a} \in \bar{K}$ is definitely a solution of \bar{p} , the “residual” polynomial over the residue field \bar{K} (this is the polynomial

with the residue map applied to the coefficients). It turns out that in \mathbb{Q}_p the converse is almost true: it is a *Henselian* valued field.

Definition 1.2.12. A valued field (K, v) is a *Henselian valued field* if whenever $p(x)$ is a polynomial over \mathcal{O}_v , and α_0 is a simple root of \bar{p} in \bar{K} , there is a unique root α of p in \mathcal{O}_v with $\bar{\alpha} = \alpha_0$ Henselian valued field

We recall that a root α of $p(x)$ is simple if it is not a root of $\frac{p(x)}{x-\alpha}$ (equivalently, of $p'(x)$).

Remark 1.2.13. Geometrically, \mathcal{O}_v can be viewed as the ring of (germs of) functions around a fixed point 0 (say, on some curve). From this point of view, \mathcal{M}_v corresponds to the ideal of functions that take the value 0 at 0, and the residue map $\alpha \mapsto \bar{\alpha}$ is evaluation of α at 0. The polynomial p over \mathcal{O}_v is viewed as a family of polynomials over a neighbourhood of 0, i.e., a family of finite sets. The residual polynomial \bar{p} then corresponds to the finite set that lies over 0. The Henselian property says that any element of this finite set can be extended to a section over the whole neighbourhood, provided the point is simple (without multiplicities). Thus, it is a form of the implicit function theorem. □

We now check that the notion is relevant to our situation, i.e., that \mathbb{Q}_p is Henselian. This is the original Hensel’s Lemma:

Proposition 1.2.14 (Hensel’s Lemma). *The field \mathbb{Q}_p is Henselian*

Proof. It is enough to show that if $q(x)$ is over \mathcal{O} and has a simple root α over $\mathbb{Z}/p^n\mathbb{Z}$, it can be lifted to a (unique) root in $\mathbb{Z}/p^{n+1}\mathbb{Z}$. Let b be any lift of α . An arbitrary lift has the form $b + cp^n$, for some c . Using Taylor expansion we obtain

$$q(b + cp^n) = q(b) + q'(b)cp^n \tag{6}$$

since higher powers of p vanish. Hence, we need to find an element c satisfying $q'(b)cp^n = -q(b)$ (in $\mathbb{Z}/p^n\mathbb{Z}$). This can be done (uniquely) since $q'(b)$ is invertible, and $q(b)$ is a multiple of p^n . □

Exercise 1.2.15. The argument can also be presented topologically, for an arbitrary valued field. Recall that a normed field is *complete* if any Cauchy sequence (sequence (a_n) satisfying $\lim_{n,m} |a_n - a_m| = 0$) has a limit. complete

- (1) Assume that the value group of a valued field is an *Archimedean group*, i.e., any element is bounded by some integer, and that K is complete. Show that K is Henselian Archimedean group
- (2) Show that \mathbb{Q}_p is complete

Definition 1.2.16. A *p-adically closed field* is a p-adic field that is Henselian p-adically closed field

Hensel’s lemma shows that \mathbb{Q}_p is p-adically closed. It is clear that p-adically closed fields are axiomatisable in the language of valued fields, as above. However, they do not have elimination of quantifiers in this language: The set of elements having an n-th root is not definable without quantifiers. It turns out that this is the only obstacle:

Theorem 1.2.17 ([3, 4, 17]). *The theory of p -adically closed fields eliminates quantifiers in the language expanded by the root predicates R_n , defined via $R_n(x) \iff \exists y(y^n = x)$ (as well as the divisibility predicates D_n from 1.2.6.)*

To prove the theorem, we begin with some preliminaries on valued fields. The Henselian property provides information about simple roots of polynomials over \mathcal{O} , inside \mathcal{O} . The following lemma is a reformulation for the more general situation.

Lemma 1.2.18. *Let K be an Henselian valued field, and let $F(x) = \sum a_i x^i$ be a polynomial over K . Assume that $b \in K$ is such that*

$$|F(b)| < \frac{\beta^2 |F'(b)|^2}{\gamma} \quad (7)$$

where $\beta = |b|$ and $\gamma = \max\{|a_i| \beta^i\}$. Then there is a unique $a \in K$ such that $F(a) = 0$ and $|a - b| < \frac{\beta^2 |F'(b)|}{\gamma}$

Proof. We first note that if the claim is true for $F(x)$, then it is true for $uF(vx)$ for any $u, v \in K$. Hence, we may assume that $b = 1$ and $\gamma = 1$, i.e., p has coefficients in \mathcal{O} . We write $a = 1 + \frac{F(1)}{F'(1)}h$, in the hope finding h for which the claim is satisfied. Using Taylor expansion, we obtain the equation

$$0 = F(a) = F\left(1 + \frac{F(1)}{F'(1)}h\right) = F(1) + F'(1)h + \frac{F''(1)}{2}h^2 + \dots \quad (8)$$

for some polynomial $g(h)$ over \mathcal{O} . Assuming $F(1) \neq 0$, we may divide by it, to obtain a polynomial equation whose coefficients of degree bigger than 1 are multiples of $\frac{F''(1)}{2F'(1)^2} \in \mathcal{M}$. Hence, the equation reduces to $\bar{h} + 1 = 0$ in \bar{K} , and -1 lifts to a unique root h . The difference between the resulting root a and the approximate root 1 is thus $|\frac{F(1)}{F'(1)}| < |F'(1)|$, as expected, and uniqueness follows from the uniqueness of h . The case $F(1) = 0$ is obtained similarly \square

The following two results are completely algebraic statements about the Galois theory of valued fields. Their proofs are omitted (at least for the moment), and can be found, e.g., in [10] or in [14]. We note first that if v is an extension of a valuation v_0 on K to a field L , and σ is an automorphism of L over K , then $v \circ \sigma$ is again a valuation extending v_0 . In other words, $\text{Aut}(L/K)$ acts on the set of valuations on L that extend v_0 .

Proposition 1.2.19. *Let (K, Γ, v) be a valued field.*

- (1) v extends to a valuation on any extension field L of K (into a value group extending Γ)
- (2) The Galois group of K acts transitively on the set of valuations on K^a extending v .
- (3) K is Henselian if and only if there is a unique extension of v to the algebraic closure of K (i.e., the valuation ring of all extensions is the same)

In particular, any field isomorphism between Henselian valued fields preserves the valuation.

The above result implies that an algebraic extension of a Henselian field induces an extension of the residue field, and of the value group. One can ask to which extent these extensions control the full algebraic extension of the valued field. In particular, if the extension is an *immediate extension*, i.e., does not induce an extension of either the residue field or the value group, is it trivial? The answer is “no” in general, but “yes” with some additional assumptions, as in the following proposition. We note that the assumption holds for p -adic fields.

immediate extension

Proposition 1.2.20. *Assume that a Henselian valued field (K, Γ, ν) has residue characteristic 0, or that it has residue characteristic $p > 0$, and is finitely ramified, in the sense that there are finitely many values between 0 and $\nu(p)$. If L is an algebraic extension of K that does not extend the value group or the residue field, then $K = L$.*

In the proof of QE for ordered abelian groups, it was useful to assume that the substructure A is closed under quantifier free definable functions (division by n). The analogous construction for valued fields is called the Henselisation.

Proposition 1.2.21. *Any valued field (K, ν) is contained in a Henselian valued field K^h , such that any map (of valued fields) from K to a Henselian field L extends uniquely to a map from K^h to L .*

Proof. Let ν_1 be an extension of ν to the algebraic closure of K . Let G be the Galois group of K , let G_d be the stabiliser of ν_1 inside G , and let K^h be the fixed field of G_d . It is easy to see that G_d is closed, hence it is the Galois group of K^h . Since G acts transitively on extensions of ν , the valuation on K^h extends uniquely to the algebraic closure, hence is Henselian by 1.2.19.

If L is a Henselian field containing K , let L_1 be the algebraic closure of K in L . Clearly, L_1 is Henselian. Again since the action of G is transitive, we may embed L_1 in the algebraic closure so that ν_1 restricts to the valuation on L_1 . Since L_1 is Henselian, this is the unique extension, so the Galois group of L_1 is a subgroup of G_d . Hence L_1 extends K^h . \square

Definition 1.2.22. Given a valued field K , the field K^h from Prop. 1.2.21 is called the *Henselisation* of K .

Henselisation

The final ingredient we require is the existence of sufficiently many roots, that can be detected via the residue field and the value group.

Lemma 1.2.23. *Let K, Γ, ν be a p -adic field. For any element $a \in K$ of valuation 0 and any $n \in \mathbb{N} \subseteq \Gamma$ there is $m \in \mathbb{Z} \subseteq K$, prime to p , with $\nu(ma - 1) \geq n$.*

In particular, if K is Henselian, for any $k > 0$ there is m as above such that ma has an k -th root.

Proof. Since K is p -adic, we have $v(ma - 1) \geq n$ if and only if $ma = 1$ in $\mathcal{O}/\mathcal{M}^n = \mathbb{Z}/p^n$. Hence we need to show this for \mathbb{Q} with the p -adic valuation, where it is clear.

For the second part, set $n = 2v(k) + 1$, and use the previous statement together with Lemma 1.2.18. \square

We are now in position to prove quantifier elimination.

Proof of Theorem 1.2.17. Once again we use 1.2.7, where we assume, by Remark 1.2.8, that M_2 is saturated. A is now a valued field, with residue field \mathbb{F}_p and value group a \mathbb{Z} -group. Since M_1 and M_2 are Henselian, they contain (canonically) the Henselisation of A , so we may assume that A is Henselian.

We now claim that we may further assume that $v(A)$ is pure in $\Gamma(A)$ (and therefore in $\Gamma(M_i)$), i.e., if $D_n(v(a))$ holds for some $a \in A$, then there is $b \in A$ with $nv(b) = v(a)$. Let n be minimal for which this fails.

In fact, we claim that for some integer i , prime to p , ia has an n -th root in M_1 . When $v(a) = 0$, this is Lemma 1.2.23. For general a , we may find $b_1 \in M_1$ with $v(b_1) = \frac{v(a)}{n}$, so that $v(\frac{a}{b_1^n}) = 0$. By the previous case, $b^n = i\frac{a}{b_1^n}$ for some $b \in M_1$ and i , so that $(bb_1)^n = ia$.

Let i be as above, and let $b \in M_1$ satisfy $b^n = ia$. We claim that $x^n - ia$ is the minimal polynomial of b over A . Otherwise, let $F(x) = \sum_{j=0}^k a_j x^j$ be the minimal polynomial. Then for some $l \neq m \leq k < n$ we have $v(a_l b^l) = v(a_m b^m)$, i.e., $(l - m)v(b) = v(\frac{a_l}{a_m}) \in v(A)$, contradicting the minimality of n .

It now follows that any extension of A inside M_2 by an n -th root of ia is isomorphic as a field to $A(b)$. Such a root indeed exists in M_2 , since ia is in $R_n(A)$. The extension is also isomorphic as a valued field, by 1.2.19.

To show the claim, it remains to show that for all $k > 1$, an element of $A(b)$ has a k -th root in M_1 if and only if it has one in M_2 . Such an element b_1 has the form $\sum_{i=0}^{n-1} a_i b^i$ for $a_i \in A$, and by a calculation as above, it has valuation $v(a_i b^i)$ for $0 \leq i < n$. Again by Lemma 1.2.23, there is an integer m (prime to p) such that mb_1 has a k -th root if and only if $a_i b^i$ does (note that m and i depend only on residues and valuations in $A(b)$ and not on the containing models M_i). Raising this term to the power n , we end up in A , where divisibility is determined by the structure A .

This concludes the proof that we may extend A to a sub-structure $A(b)$, in which $v(b) = \frac{v(a)}{n}$. Applying this procedure repeatedly, we obtain the purity claim.

Given the purity, any $b \in M_1$ not in A is transcendental over A by 1.2.20. Assume that $\gamma_1 = v(b) \notin v(A)$. We may find $\gamma_2 \in \Gamma(M_2)$ having the same type over $v(A)$ as γ_1 (by quantifier elimination in \mathbb{Z} -groups). Since $v(A)$ is pure in each of the groups, the valuation of every rational function in b is determined by setting $v(b) = \gamma_2$ in M_2 . These conditions do not interact with the root conditions, and therefore they can also be satisfied.

Finally, assume that $v(b) \in v(A)$. Dividing by a suitable element of A , we may assume $v(b) = 0$. Since b is transcendental, there are no polynomial equations to satisfy, and other kinds of formulas can be realised by approximating b with sufficient precision, which can be done by Lemma 1.2.23. \square

Corollary 1.2.24. *The theory of p -adically closed fields is the model completion of the theory of p -adic fields. In particular, it is the complete theory of \mathbb{Q}_p .*

We recall that a theory \mathcal{T} is the *model completion* of a universal theory \mathcal{T}_0 if any model of \mathcal{T}_0 extends to a model of \mathcal{T} , and \mathcal{T} eliminates quantifiers. model completion

Proof. If K is a p -adic field, its Henselisation has the same residue field and value group, so is a p -adically closed field into which K embeds.

By quantifier elimination, every formula, and thus every sentence is equivalent to a quantifier free one. Since every model of the theory contains the Henselisation of \mathbb{Q} with the p -adic valuation, every such sentence is decided. Hence the theory is complete, and we have already seen that \mathbb{Q}_p is a model. \square

1.3. Definable subsets of p -adically closed fields. The quantifier elimination result allows us to start analysing the collection of definable sets in powers of K (“definable” will now mean: with parameters from K). We recall that an arbitrary valuation on a field K induces a topology on K , generated by the open balls $\mathbf{O}_\gamma(\alpha) = \{x \in K : |x| < \gamma\}$ for $\alpha \in K$ and $\gamma \in \Gamma$. We take the product topology on powers of K . By explicit inspection, it is easy to see, then, that an arbitrary definable subset X is a finite union of subsets X_i , each an intersection of an open subset and a Zariski-closed subset (i.e., a subset given by a finite number of polynomial equations). In particular, we have

Corollary 1.3.1. *Let A be a sub-structure of a model M of \mathcal{T} .*

- (1) *The (model theoretic) algebraic closure of A consists of the field theoretic algebraic closure in the field sort, and the divisible hull of $\Gamma(A)$ in the value group.*
- (2) *If $\Gamma(A)$ is contained in the definable closure of $v(A)$, then the algebraic closure of A is an elementary sub-model of \mathcal{T} .*
- (3) *The definable closure of A coincides with the algebraic closure. In particular, if $\gamma \in \Gamma$ is definable over $\mathbf{VF}(A)$, then $\gamma = \frac{v(\alpha)}{m}$ for some $m \in \mathbb{N}$ and $\alpha \in \mathbf{VF}(A)$.*

Proof. (1) A finite A -definable subset of \mathbf{VF} is necessarily a subset of a proper Zariski-closed subset of \mathbf{VF} , hence contained in the field theoretic (relative) algebraic closure.
 (2) Let B be the algebraic closure. Then B is Henselian, and its value group is pure in the value group of M . Also, it is closed under roots, so the \mathbf{R}_i are interpreted correctly. Hence it is a submodel, which is elementary by quantifier elimination.

- (3) Let B be the algebraic closure of A , and let C be the fixed subfield of B under $\text{Aut}(B/A)$. Then C contains the Henselisation of A (since B is Henselian, and the Henselisation embeds uniquely in it), hence is itself Henselian. Hence, by Prop. 1.2.20, it suffices to prove that $v(\mathbf{VF}(C)) = v(\mathbf{VF}(B))$, and for that, it suffices to prove that C is closed under roots.

Assume that $b^n = a \in C$ for some $b \in B$. By Lemma 1.2.23, for any $k > 0$ there is $m \in \mathbb{Z}$ such that mb has a k -th root. If b_1 satisfies the same type over C , it follows in particular that mb_1 has a k -th root as well, hence so does $\frac{mb}{mb_1} = \frac{b}{b_1}$, a root of 1. Thus we obtain a root of 1 that has a k -th root for all k . The following lemma implies that the root of unity is equal to 1, i.e., that $b = b_1$. Thus b is definable over C .

□

Lemma 1.3.2. *A model of \mathcal{T} contains only a finite number of roots of unity*

Proof. Exercise, using Hensel's Lemma

□

Our next goal is to analyse in detail the definable subsets \mathbf{VF} (i.e., sets definable with one free \mathbf{VF} -variable). This is the main technical ingredient in Denef's computation of the integrals.

Proposition 1.3.3. *Any definable subset of \mathbf{VF}^n is a boolean combination of sets of the form $\mathbf{R}_n(f_i(\bar{x}))$, where f_i are polynomials. Any definable subset of \mathbf{VF} is a disjoint union of a finite set, and sets defined by formulas of the form $\bigwedge_i \mathbf{R}_n(f_i(x))$ (where n is independent of i).*

Proof. By quantifier elimination, it suffices to consider atomic formulas. Each atomic formula not of the required form is equivalent to a polynomial equation, or to $\lambda(v(f_1), \dots, v(f_n)) > 0$, where $\lambda : \mathbf{\Gamma}^n \rightarrow \mathbf{\Gamma}$ is a linear function $\lambda(\gamma_1, \dots, \gamma_n) = \sum m_i \gamma_i + \gamma$ with $m_i \in \mathbb{Q}$ and $\gamma \in \Gamma$. A polynomial equation $g(x) = 0$ is equivalent to $\neg \mathbf{R}_2(g^2(x))$, so we need only to deal with the second kind.

By clearing denominators, we may assume all $m_i \in \mathbb{Z}$. We may also multiply by a sufficiently large integer, and assume that $\gamma = v(a)$ for some constant a . We then have that the formula is equivalent to $v(F) > v(G)$ for suitable polynomials F and G .

We now claim that the relation $v(x) > v(y)$ is equivalent to $\mathbf{R}_2(px^2 + y^2)$ if $p > 2$ (if $p = 2$, replace p by p^3). Indeed, if $v(x) > v(y)$, then $v(px^2) = 2v(x) - 1 > 2v(y) = v(y^2)$, hence $v(px^2 + y^2) = v(y^2)$. It follows that $px^2 + y^2$ has a square root if and only if $1 + p\frac{x^2}{y^2}$ does, which it does by Hensel's Lemma. Conversely, if $v(x) \leq v(y)$, then $v(px^2 + y^2) = v(px^2) = 2v(x) - 1$, hence it cannot have a square root.

To prove the second part, we note that $\neg \mathbf{R}_n(x)$ is equivalent to the (disjoint) disjunction of $\mathbf{R}_n(d_i x)$, where d_i are integer representative for the elements of $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times n}$. Also, for each m , $\mathbf{R}_k(x)$ is equivalent to a conjunction

of formulas $\mathbf{R}_{km}(d_i x^m)$ for suitable elements d_i (namely, d_i are equivalent, in $\mathbb{Q}_p^x / \mathbb{Q}_p^{x^m}$, to certain roots of unity). \square

We recall that the ball with centre c , of (valuative) radius γ , is the definable set $B_\gamma(c) = \{x \in \mathbf{VF} : v(x - c) \geq \gamma\}$ (this is, more precisely, a closed ball, but since the value group is discrete, any open ball is closed as well). We note that any element of $B_\gamma(c)$ is a centre for it. The radius, however, is uniquely determined.

A *Swiss cheese* is a ball with a finite number of proper sub-balls removed. Any centre of the containing ball is also called a centre of the Swiss cheese (thus, a centre need not belong to the swiss cheese). We remark that the presentation as a difference of balls is a property of the formula, rather than the set it defines: a definable set can be a Swiss cheese in many different ways.

Swiss cheese

Proposition 1.3.4. *Let K be a p -adically closed field. For any polynomial $f(t) \in K[t]$ and natural number $n > 0$, there is a partition of \mathbf{VF} into finitely many swiss cheeses \mathbf{A} , such that on \mathbf{A} we have for some centre c of \mathbf{A} , $f(t) = hu(t)^n(t - c)^e$, with $|u(t)| = 1$ for all t , and h independent of t .*

Remark 1.3.5. The statement clearly implies that u is definable as well \square

To prove the statement, we will pass to a finite extension of K . If E is such an extension, then we may obviously write $E = V + K$ as a vector space, where V is a K -subspace of E , with $V \cap K = 0$. We would like to obtain such a decomposition as *valued* vector spaces:

Lemma 1.3.6. *Assume that E is a finite extension of a Henselian valued field K of characteristic 0, such that the residue field extension is separable. Then there is a K -vector space decomposition $E = K + V$ of E , such that $v(x + y) = \min(v(x), v(y))$ for $x \in V$ and $y \in K$.*

For the proof of the lemma, we recall that the *trace* $\text{Tr}(e)$ of an element $e \in E$ is its trace when viewed as the linear map from E to itself given by multiplication by e . When E is an unramified extension of K , the (residue field) trace of an element \bar{e} , for $e \in \mathcal{O}_E$, coincides with the residue $\text{Tr}(\bar{e})$. Since the residue field extension is separable, the trace map is surjective there, and we may find an element $\bar{a} \in \bar{E}$ with trace 1. Lifting it to an element $a \in E$, we thus obtain an element with $\text{Tr}(a) = 1$ and $v(a) = 0$ (if the degree N of the extension is not divisible by p , we may simply take $a = \frac{1}{N}$).

Proof of Lemma 1.3.6. Since assumptions remain valid for intermediate extensions, we may assume that E is either unramified (no value group extension) or totally ramified (no residue field extension). The second case is easy, and is left as an exercise. We thus assume that E is unramified.

In this case, let a be as above, and let $\pi : E \rightarrow E$ be given by $\pi(x) = \text{Tr}(ax)$. Clearly, π takes values in K and is the identity on K , so setting $V = \ker(\pi)$, we have a direct sum decomposition in which π is the projection to K .

To prove the condition in the lemma, we may assume $v(x) = v(y) = 0$, and therefore that $y = 1$ and $\text{Tr}(ax) = 0$, and we need to show that the residue of x is not 1. If it was 1, we would have $x = 1 + u$ with $v(u) > 0$, and therefore $\text{Tr}(ax) = \text{Tr}(a) + \text{Tr}(au)$. Since

$$v(\text{Tr}(au)) \geq v(au) = v(a) + v(u) > v(a) = 0$$

we have $v(\text{Tr}(ax)) = v(\text{Tr}(a)) = 0$, contradicting that $\text{Tr}(ax) = 0$. \square

Proof of Prop. 1.3.4. We may assume that f is irreducible. Write $f(t) = (t - s_1) \dots (t - s_d)$ with s_i in the splitting field E of f . Let $E = V + K$ as in Lemma 1.3.6, and let $c_i \in K$ be the projection of s_i to K . Then for all $x \in K$ we have $v(x + s_i - c_i) = \min(v(x), \gamma_i)$, where $\gamma_i = v(s_i - c_i)$. Let $B_i = \{t \in \mathbf{VF} : v(t - c_i) \geq \gamma_i\}$ be the (closed) ball of radius γ_i around c_i . For elements $t \in K$ we have

$$v(t - s_i) = v(t - c_i + c_i - s_i) = \min(v(t - c_i), \gamma_i) = \begin{cases} \gamma_i & t \in B_i, \\ v(t - c_i) & t \notin B_i. \end{cases} \quad (9)$$

For $I \subseteq \{1, \dots, d\}$, let $A_I = \bigcap_{i \in I} B_i \setminus \bigcup_{i \notin I} B_i$ and $\gamma_I = \sum_{i \in I} \gamma_i$. Clearly, the A_I form a partition of \mathbf{VF} , and each A_I is a swiss cheese. On A_I we have

$$v(f(t)) = \sum_{i=1}^d v(t - s_i) = \gamma_I + v\left(\prod_{i \notin I} (t - c_i)\right) \quad (10)$$

Let h_I be an element of K with valuation γ_I , and let $f_I(t) = \prod_{i \notin I} (t - c_i)$. Then on A_I , f and $h_I f_I$ have the same valuation, hence their quotient $u_I(t)$ has valuation 0 there. By modifying h_I and partitioning further (by smaller balls), we may assume that $u_I(t)$ has residue 1 in sufficiently fine quotient \mathbb{Z}_p/p^m , so that u_I is an n -th power. \square

With some more detailed analysis, we may obtain the following finer version: Swiss cheeses are replaced by annuli, and the partition can be done simultaneously for several polynomials. We omit the proof, cf. [6, Theorem 7.3].

Proposition 1.3.7. *Given a finite number of polynomials $f_i(t)$ and a natural number n , there is a finite partition of \mathbf{VF} into annuli $\mathbf{A} = \mathbf{A}(c_j)$, such that the restriction of each f_i to \mathbf{A} has the form $h_i u_i(t)^n (t - c_j)^{e_i}$ for some constant h_i , natural number e_i , and unit-valued definable function u_i .*

We call a definable subset of \mathbf{VF} a *simple set* if it is finite, or is the intersection of an annulus centred at c with a set of the form $\mathbf{R}_n(d(t - c))$ for some integer d .

Proposition 1.3.8. *Let $\mathbf{X} \subseteq \mathbf{VF}$ be a definable set (in one variable). Then \mathbf{X} is a finite disjoint union of simple sets. If $r : \mathbf{X} \rightarrow \Gamma$ is definable, the pieces can be chosen so that r has the form $r(t) = qv(t - c) + \gamma$, where q is rational, c is a centre, and $\gamma \in \Gamma$. each.*

Proof. By 1.3.3, \mathbf{X} is a disjoint union of definable sets, each of the form

$$\bigwedge_i \mathbf{R}_n(f_i(x)) \wedge \bigwedge_j g_j(x) = 0 \quad (11)$$

for a fixed n , where f_i and g_j are polynomials over $\mathbf{VF}(A)$. Hence, we may assume that \mathbf{X} itself is of this form. We may further assume that no g_j appears, since the set defined by them is finite (hence simple). Also, by 1.3.1, we may assume that $r(t) = qv(\frac{r_1(t)}{r_2(t)})$, where r_i are polynomials, and q is rational.

According to 1.3.7, there is a partition of \mathbf{VF} to a finite number of annuli $\mathbf{A}_{\alpha,\beta}(c)$, on which every $f_i(t)$ and r_i has the form $h_i u_i(t)^n (t-c)^{e_i}$ for some natural e_i . We may therefore restrict attention to one such annulus. Since we are interested in the class of this expression in $\mathbf{VF}^x/\mathbf{VF}^{x^n}$, we may ignore $u_i(t)$, and assume that h_i are natural numbers. The formula is then equivalent to a conjunction of formulas of the form $\mathbf{R}_n(d_i(t-c))$, where $d_i \in \mathbb{N}$ (namely, d_i represents the inverse of the e_i -th root the class of h_i in the above group). These formulas are clearly inconsistent (or equal), so in fact, only one appears. \square

1.4. Constructible functions. From now on, we work in the model \mathbb{Q}_p . We identify a definable set in the theory of p -adically closed fields with the set of \mathbb{Q}_p -points it determines. In particular, $\Gamma = \Gamma(\mathbb{Q}_p) = \mathbb{Z}$. Our goal is to define a class of functions, the constructible functions, which is closed under integration and includes the functions we are interested in. We begin this by considering functions on the value group only.

- Definition 1.4.1.**
- (1) A subset $\mathbf{X} \subseteq \Gamma^n$ is *bounded* if for some $a \in \Gamma$ we have $\alpha_i \geq a$ for all $(\alpha_1, \dots, \alpha_n) \in \mathbf{X}$. bounded
 - (2) The *support* of a function $f : \Gamma^n \rightarrow \mathbb{Q}$ is the set of all x with $f(x) \neq 0$. We say that f is *compactly supported* if the support of f is bounded. support
compactly supported
 - (3) For $f = f(\bar{x}, t) : \Gamma^{n+1} \rightarrow \mathbb{Q}$ compactly supported, we define $\Sigma(f) = \Sigma_t(f) : \Gamma^n \rightarrow \mathbb{Q}$ via $\Sigma_t(f)(x) = \sum_{t \in \Gamma} f(x, t)p^{-t}$ (in general, this need not converge)

Remark 1.4.2. These definitions make more sense when using the multiplicative version of Γ . In particular, the summation defined above corresponds to integrating a function along the t -fibres using the *usual* Lebesgue measure. \square

Let \mathbf{Y} be a bounded definable subset of Γ^m , and let $r : \mathbf{Y} \rightarrow \Gamma^n$ be a bounded function with finite fibres. To this datum, we may assign a function $F_{\mathbf{Y},r} : \Gamma^n \rightarrow \mathbb{Q}$, given by $F_{\mathbf{Y},r}(x) = \#r^{-1}(x)$ (of course, this function actually takes values in \mathbb{N} , but we view them as functions into \mathbb{Q}). We sometimes omit r and write the fibre $r^{-1}(x)$ as \mathbf{Y}_x .

Definition 1.4.3. A function of the form $F_{\mathbf{Y},r}$, where $r : \mathbf{Y} \rightarrow \Gamma^n$ is bounded definable is called *basic constructible*. The group $\mathcal{C}(\Gamma^n)$ of (compactly sup- basic constructible

ported) *constructible functions* on Γ^n is the (additive) group of functions from Γ^n to \mathbb{Q} generated by functions of the form $\Sigma(F)$, where F is basic constructible (on Γ^{n+1}).

constructible functions

More generally, we denote by $\mathcal{C}(\mathbf{X})$, for $\mathbf{X} \subseteq \Gamma^n$, the subgroup of $\mathcal{C}(\Gamma^n)$ consisting of functions supported on \mathbf{X} . We will see below that the definition makes sense, i.e., that the series defining $\Sigma(F)$ converges.

- Proposition 1.4.4.** (1) *If F is constructible on Γ^n , and $g : \Gamma^m \rightarrow \Gamma^n$ is bounded definable, then $F \circ g$ is constructible on Γ^m . In particular, any compactly supported definable function is constructible*
(2) *Any basic constructible function is constructible*
(3) *If F is a constructible function on Γ^{n+1} , then $\Sigma(F)$ is constructible (on Γ^n)*
(4) *The product of constructible functions is constructible*

We remark that we will describe below the definable functions in Γ , and in particular those which are bounded.

Proof. (1) This easily reduces to the analogous statement for basic constructible, $F = F_{\mathbf{Y},r}$. Let

$$\mathbf{Z} = g^*(\mathbf{Y}) = \{(x, y) \in \Gamma^m \times \mathbf{Y} : g(x) = r(y)\} \quad (12)$$

with $s : \mathbf{Z} \rightarrow \Gamma^m$ the projection to Γ^m . Then for all $x \in \Gamma^m$, $s^{-1}(x) = r^{-1}(g(x))$, from which the statement follows. The second part follows since the identity function on each bounded set is easily seen to be constructible.

- (2) Given $r : \mathbf{Y} \rightarrow \Gamma^n$, let $s : \mathbf{Y} \rightarrow \Gamma^{n+1}$ be given by $s(y) = (r(y), 0)$. Clearly, $F_{\mathbf{Y},r} = \Sigma(F_{\mathbf{Y},s})$.
(3) It suffices to prove this for generators. Let $r = (r_{\mathbf{X}}, r_1, r_2) : \mathbf{Y} \rightarrow \mathbf{X} \times \Gamma \times \Gamma$, and define $s : \mathbf{Y} \rightarrow \mathbf{X} \times \Gamma$ by $s(y) = (r_{\mathbf{X}}(y), r_1(y) + r_2(y))$. We note that $s^{-1}(x, k) = \coprod_{i+j=k} r^{-1}(x, i, j)$, and this set is finite, since for each k , only finitely many such i, j determine a non-empty fibre. We therefore have

$$\begin{aligned} \Sigma(\Sigma(F_{\mathbf{Y},r}))(\mathbf{x}) &= \Sigma_i(\Sigma(F_{\mathbf{Y},r})(\mathbf{x}, i)p^{-i}) = \\ &= \Sigma_i \Sigma_j \# r^{-1}(\mathbf{x}, i, j) p^{-j-i} = \Sigma_k p^{-k} \Sigma_{i+j=k} \# r^{-1}(\mathbf{x}, i, j) = \\ &= \Sigma_k p^{-k} \# s^{-1}(\mathbf{x}, k) = \Sigma(F_{\mathbf{Y},s})(\mathbf{x}) \end{aligned} \quad (13)$$

- (4) Again it suffices to prove this for generators. If $F, G : \mathbf{X} \times \Gamma \rightarrow \mathbb{Q}$ are two compactly supported functions, and $H : \mathbf{X} \times \Gamma \times \Gamma \rightarrow \mathbb{Q}$ is defined by $H(x, i, j) = F(x, i)G(x, j)$, it is easy to see that $\Sigma(F)\Sigma(G) = \Sigma(\Sigma(H))$. Therefore, it follows from the previous part that it suffices to show that if $F = F_{\mathbf{Y},r}$ and $G = F_{\mathbf{Z},s}$ are basic constructible, then so is H . Letting $\mathbf{W} = \mathbf{Y} \times_{\mathbf{X}} \mathbf{Z} = \{(y, z) \in \mathbf{Y} \times \mathbf{Z} : r_{\mathbf{X}}(y) = s_{\mathbf{X}}(z)\}$ (where $r_{\mathbf{X}}$ is the \mathbf{X} component of r), and $t : \mathbf{W} \rightarrow \mathbf{X} \times \Gamma \times \Gamma$ be given $t(y, z) = (r_{\mathbf{X}}(y), r_{\Gamma}(y), s_{\Gamma}(z))$, one easily sees that $H = F_{\mathbf{W},t}$. \square

We now extend the definition to general definable sets in the theory of \mathbb{Q}_p .

Definition 1.4.5. Let \mathbf{X} be a definable set in the theory of \mathbb{Q}_p . The space $\mathcal{C}(\mathbf{X})$ of *constructible functions* on \mathbf{X} is the \mathbb{Q} -vector space generated by functions of the form $F \circ \alpha$, where $\alpha : \mathbf{X} \rightarrow \Gamma^n$ is a definable function, and $F : \Gamma^n \rightarrow \mathbb{Q}$ is constructible (in the previous sense)

constructible functions

We note that by 1.4.4, this definition coincides with the previous one when $\mathbf{X} \subseteq \Gamma^m$. We also note that if \mathbf{X} and \mathbf{Y} are disjoint, then $\mathcal{C}(\mathbf{X} \cup \mathbf{Y}) = \mathcal{C}(\mathbf{X}) \times \mathcal{C}(\mathbf{Y})$, i.e., a piecewise-constructible function is constructible.

The following is a version of Denef's theorem.

Theorem 1.4.6. Let $\mathbf{Z} \subseteq \mathbf{X} \times \mathbf{Y}$ be a bounded definable set in the theory of \mathbb{Q}_p , and let F be a constructible function on \mathbf{Z} . Then the function

$$G : \mathbf{Y} \rightarrow \mathbb{Q}, \quad G(y) = \int_{\mathbf{Z}_y} F(x, y) dx \quad (14)$$

is constructible

We remark that \mathbf{Z} in this theorem may have components in Γ as well as \mathbf{VF} , in which case we interpret \int as Σ , in the sense of Def. 1.4.1.

Proof. By definition, there is a constructible function $G : \Gamma^m \rightarrow \mathbb{Q}$, and a definable function $\alpha : \mathbf{X} \times \mathbf{Y} \rightarrow \Gamma^m$ such that $F = G \circ \alpha$. By induction (and Fubini), it suffices to prove this when $\mathbf{X} = \mathbf{VF}$ or $\mathbf{X} = \Gamma$.

Assume $\mathbf{X} = \mathbf{VF}$. By 1.3.8, we may partition \mathbf{Z}_y into simple (y -definable) subsets $\mathbf{W}(c(y))$, on which we have $\alpha_i(x, y) = e_i v(x - c(y)) + d_i(y)$ for rational e_i and integer $d_i(y)$. The number of such pieces and the numbers e_i depend on y , but (by compactness) are constant on each of the pieces of a finite definable partition of \mathbf{Y} . Since a piecewise-constructible function is constructible, we may restrict attention to one of these pieces. Since integration is additive on disjoint unions, we may therefore assume that \mathbf{Z}_y is just one such piece. Furthermore, since integration is translation-invariant, we may assume that $c = 0$, and again by additivity, we may assume that the annulus that occurs in \mathbf{W} is, in fact, a ball. Also, we may assume $d_i = 0$ and $e_i = 1$ by making a definable change of variable in G . We are thus facing the integral

$$\int_{|x| \leq p^{-\gamma(y)}, \mathbf{R}_n(bx)} G(v(x)) = \sum_{i \geq \gamma(y)} \mu(\{x : v(x) = i, \mathbf{R}_n(bx)\}) G(i) \quad (15)$$

We now recall that for any natural n and l , the set $\mathbf{R}_{n,l} = v^{-1}(l) \cap \mathbf{R}_n$ is non-empty precisely if l is divisible by n , and given such n, l , and $z \in \mathbf{R}_{n,l}$, the map $x \mapsto zx$ determines a bijection from $\mathbf{R}_{n,0}$ to $\mathbf{R}_{n,l}$, scaling the measure by p^{-l} . Thus, setting $r = \mu(\mathbf{R}_{n,0})$ (a rational number), we obtain that the

integral above is given by

$$r\left(1 - \frac{1}{p}\right) \sum_i p^{-i} H(i, \gamma(y)) \quad (16)$$

where

$$H(i, j) = \begin{cases} G(i) & i \geq j, D_n(i), \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

Clearly, H is constructible, and our integral has the form $\Sigma(H)(\gamma(y))$. By Prop. 1.4.4, $\Sigma(H)$ is constructible, so by definition, the integral is a constructible function as well. The case $\mathbf{X} = \mathbf{\Gamma}$ follows again from 1.4.4. \square

1.5. Towards motivic integration. The goal of this subsection is to explain the transition from the classical p -adic integrals that occur in the present section, to the computation of Grothendieck semi-rings that occurs in the rest of these notes, along with some motivation and history. We follow (initially) the notes [16]. Some more detailed overviews (of the flavours that we will not discuss further) appear in [6] and [19].

1.5.1. The topological zeta function. Going back to the situation in the beginning of the section, we mentioned that the Igusa zeta function associated to a polynomial F can be computed in terms of an embedded resolution of singularities of the variety $\mathbf{X} \subseteq \mathbb{A}^n$ defined by F . This expression takes the following form: The resolution \mathbf{Y} is a certain map $\pi : \mathbf{Y} \rightarrow \mathbb{A}^n$, such that the irreducible components $\{\mathbf{E}_i : i \in I\}$ of the inverse image $\pi^{-1}(\mathbf{X})$ of \mathbf{X} are smooth. To each component \mathbf{E}_i one associates certain integers (N_i, ν_i) related to the way \mathbf{E}_i is embedded in \mathbf{Y} (namely, N_i is the multiplicity of $F \circ \pi$ on \mathbf{E}_i , and ν_i is the multiplicity of the relative sheaf of differentials $\Omega_{\mathbf{Y}/\mathbb{A}^n}$ over \mathbf{E}_i). Then using the change of variable formula for integration, and a computation similar to the one in 1.1.2, Denef showed that (with certain assumptions on the reduction of the resolution) the following formula holds:

$$\{E:zeta\} \quad Z_F(s) = q^{-n} \sum_{J \subseteq I} \#E_J(k) \prod_{j \in J} \frac{(q-1)q^{-N_j s - \nu_j}}{1 - q^{-N_j s - \nu_j}} \quad (18)$$

where k is the residue field (of cardinality q), and for a subset J of I , $E_J = \bigcap_{j \in J} E_j \setminus \bigcup_{j \notin J} E_j$.

This formula can be viewed as an aid for computing the zeta function. However, it can also be viewed as providing geometric information about \mathbf{X} : the resolution of \mathbf{X} , and in particular the number of components \mathbf{E}_i , and the numbers N_i and ν_i are not unique. However, the expression on the right of (18) which appears to depend on them, is in fact independent, and depends only on F (since the left hand side does not depend on the resolution). Furthermore, the expression on the right is uniform: It is expressed in terms of the number of points in certain varieties (note that $q = \#\mathbb{A}^1(k)$). Thus the situation is that we have two equal quantities associated to F : the

left hand side, which is independent of the resolution, and the right hand side, which is almost independent of the valued field.

Since the coefficients of F are integers, we may also view F as a polynomial over \mathbb{C} . In this case, $\mathbf{X}(\mathbb{C})$ is a complex analytic space, and we may be interested in its topological properties. In particular, one may ask whether it is possible to write a topological zeta function, similar to the one on the right hand side of (18), which will not depend on the resolution. Evaluating this expression over \mathbb{C} poses the problem that the number of complex points of a variety is often infinite, but Denef and Loeser realised that the correct heuristic here is to “take the limit as $q = p^r$ goes to 1 ”¹, to obtain the following result (using the terminology above).

Theorem 1.5.2 ([5]). *The topological zeta function Z_F^{top} of F , defined as*

$$Z_F^{\text{top}}(s) = \sum_{J \subseteq I} \frac{\chi(E_J(\mathbb{C}))}{\prod_{j \in J} (N_j s + \nu_j)} \tag{19}$$

where $\chi(\mathbf{Y})$ is the compactly supported Euler characteristic, does not depend on the resolution

While the heuristic leads to the definition of the topological zeta function, it does not provide a proof. Following the ideas from the p -adic situation, one would hope to express the function as an integral, perhaps over $\mathbb{C}[[t]]^n$, so that the dependence on the resolution disappears. However, $\mathbb{C}[[t]]$ is not locally compact, and so no classically defined integration theory exists. In an unpublished note, Kontsevich suggested that there should exist a generalised integration theory, where the integrals take values in a suitably defined Grothendieck ring of varieties. This idea was implemented by Denef and Loeser, under the name “motivic integration”. Their theory in particular provides a proof of the above theorem (though this was not the original proof).

1.5.3. *The Grothendieck ring of varieties.* Above, we considered two operations i on varieties \mathbf{X} defined over \mathbb{Z} : Counting points in a finite field, and computing the Euler characteristic of the space of complex points. The two operations have the following in common: They are invariant under (algebraic) isomorphisms, they are additive: $i(\mathbf{X} \cup \mathbf{Y}) = i(\mathbf{X}) + i(\mathbf{Y})$ when \mathbf{X} and \mathbf{Y} are disjoint, and they are multiplicative: $i(\mathbf{X} \times \mathbf{Y}) = i(\mathbf{X})i(\mathbf{Y})$. An operation satisfying these properties is called an *additive invariant* (multiplicativity is sometimes ignored).

additive invariant

It makes sense to consider additive invariants on varieties over a field \mathbb{k} taking values in an arbitrary commutative ring (in fact, semi-ring), and among these, there is a universal one $\mathbf{X} \mapsto [\mathbf{X}] \in K(\mathcal{V}\text{ar}_{\mathbb{k}})$, where $K(\mathcal{V}\text{ar}_{\mathbb{k}})$, the *Grothendieck ring of varieties* is defined as the ring generated freely by the isomorphism classes of varieties over \mathbb{k} , subject to the additivity and

$K(\mathcal{V}\text{ar}_{\mathbb{k}})$
Grothendieck ring of varieties

¹We will slightly expand on this heuristic in §4.5

multiplicativity relations above. We note that though this invariant is very easy to define, it is very hard to understand. Indeed, defining an additive invariant on varieties is equivalent to defining a ring homomorphism from $K(\mathcal{V}\text{ar}_{\mathbb{k}})$, so understanding this ring amounts to understanding all additive invariants of algebraic varieties.

As mentioned above, both the classical and the topological zeta function can be expressed completely expressed in terms of additive invariants. The idea is thus to replace values of particular additive invariants, by the universal one, and consider integrals that take values in the Grothendieck ring of varieties. Since the number of points of \mathbb{A}^1 occurs inverted in the classical zeta function, one in fact expects the class \mathbb{L} of \mathbb{A}^1 to be inverted, i.e., the candidate for values is $K(\mathcal{V}\text{ar}_{\mathbb{k}})[\mathbb{L}^{-1}]$ (this is not visible in the topological version since the Euler characteristic of \mathbb{A}^1 is 1).

1.5.4. *Geometric motivic integration.* Having discussed the range of the motivic measure, we now briefly discuss the domain. We recall that the original plan was to integrate subsets of $\mathbb{A}^n(\mathbb{C}((t)))$, where $\mathbb{C}((t))$ is the field of Laurent series. This is essentially what is done, but to describe the “measurable” subsets, a more algebraic description is required.

Let k be a field. The *arc space* $\mathcal{L}(\mathbf{X})$ of a (quasi-projective) scheme \mathbf{X} over k is determined by the property that $\mathcal{L}(\mathbf{X})(A) = \mathbf{X}(A[[t]])$ (functorially) for algebras A over k . The geometric motivic measure is defined on certain subsets of $\mathcal{L}(\mathbf{X})$ called *cylinders*, and takes values in $K(\mathcal{V}\text{ar}_{\mathbb{k}})[\mathbb{L}^{-1}]$ (in fact, the class of measurable sets is enlarged by taking a certain completion \mathcal{M} of this ring). We omit the details, but mention that $\mathcal{L}(\mathbf{X})$ itself is a cylinder, and if \mathbf{X} is smooth of pure dimension d , then $\mu(\mathcal{L}(\mathbf{X})) = [\mathbf{X}]\mathbb{L}^{-d}$.

Since the measure is defined on subsets of $\mathcal{L}(\mathbf{X})$, and takes values in \mathcal{M} , we expect to be able to integrate certain functions on $\mathcal{L}(\mathbf{X})$ that take values in \mathcal{M} . By analogy with the p -adic case, one defines, for functions $\alpha : \mathcal{L}(\mathbf{X}) \rightarrow \mathbb{N}$,

$$\int_{\mathcal{L}(\mathbf{X})} \mathbb{L}^{-\alpha} = \sum_{i \in \mathbb{N}} \mu(\alpha^{-1}(i)) \mathbb{L}^{-i} \quad (20)$$

assuming that the right hand side makes sense as an element of \mathcal{M} (in particular, each $\alpha^{-1}(i)$ should be measurable; the completion on \mathcal{M} is such that the integral exists for “nice” functions α).

In particular, given a polynomial F as before, setting $\alpha(x) = v(F(x))$ for $x \in \mathbf{X}(A[[t]]) = \mathcal{L}(\mathbf{X})(A)$ (note that the value is independent of A , and is therefore genuinely a function on the arc space), and replacing \mathbb{L} by \mathbb{L}^{-s} , one obtains the motivic analogue Z_F^{mot} of the zeta function. A motivic analogue of the change of variables formula for integration allowed Denef and Loeser to repeat the calculation of the classical zeta function, to obtain the formula

$$Z_F^{\text{mot}}(s) = \mathbb{L}^{-n} \sum_{J \subseteq I} [E_J] \prod_{j \in J} \frac{(\mathbb{L} - 1) \mathbb{L}^{-v_j - s N_j}}{1 - \mathbb{L}^{v_j - s N_j}} \quad (21)$$

where notation is as above. The formula for the classical zeta function occurs as a specialisation of this formula, and specialising to the Euler characteristic one obtains the topological zeta function, proving the theorem of Denef and Loeser cited above.

1.5.5. *From Denef–Loeser to Hrushovski–Kazhdan.* Geometric motivic integration can be viewed as a uniformity result: the expression for a p -adic integral depends on the residue field in a way that can be expressed uniformly via varieties over the residue field (and the expressions continue to make sense when the residue field is not finite). In other words, the expression remains the same when passing to *unramified* extensions of the valued field. However, the uniformity does not extend to ramified extensions. One advantage of the theory of [12] is that expressions are obtained which are uniformly valid for all (0-characteristic) valued fields.

The value group \mathbb{Z} of a local field occurs in the integration theory as the index set for summation: As we saw in the p -adic case, an integral is given as an infinite sum over the integer tuples that lie in a definable subset of Γ . The idea is that just like the set of points (in the residue field) of a variety is replaced by the class of the variety, the set of indices should be replaced by the definable set in Γ that determines it. In other words, we expect to have a motivic measure that takes value in Grothendieck ring that combines varieties and definable sets in the value group Γ .

Another modification occurs in the domain of the measure. Rather than viewing measurable sets as (limits of) objects from algebraic geometry, the “measurable sets” are now definable sets in the theory $\mathcal{ACVF}_{0,0}$ of algebraically closed fields of equal characteristic 0. Since a measure is itself a kind of an additive invariant, we are led to consider Grothendieck (semi-) rings of theories.

1.5.6. *Grothendieck rings of first order theories.* Even if we don’t know how to define a measure on definable sets, we know some of its properties: Since it should be analogous to Haar measure, it should be invariant under translations, and it should be additive with respect to disjoint unions, and multiplicative on Cartesian products (Fubini). These properties can be formalised via the *Grothendieck ring of definable sets*.

This makes sense for any first order theory \mathcal{T} . Given a sub-category \mathcal{C} of the category of definable sets and definable maps between them, one defines $K_+(\mathcal{C})$, the Grothendieck semi-ring of \mathcal{C} , to be the semi-ring generated freely by the isomorphism classes in \mathcal{C} , subject to the relations $[X \cap Y] + [X \cup Y] = [X] + [Y]$ and $[X \times Y] = [X] \cdot [Y]$ (the Grothendieck ring is defined analogously, but contains less information, as explained in 3.4. Working with semi-rings is another advantage of this flavour).

With this terminology, the task of defining a “measure” on the objects of \mathcal{C} , for which the morphisms in \mathcal{C} are “measure preserving” becomes equivalent to defining a homomorphism on $K_+(\mathcal{C})$. There are several instances of this in [12], where \mathcal{T} is $\mathcal{ACVF}_{0,0}$. As explained above, the measure takes

Grothendieck ring of definable sets

values in a combination of the Grothendieck semi-rings of varieties and of the value group, both of which are sorts in $\mathcal{ACVF}_{0,0}$. Thus, the problem becomes about computing the relation between Grothendieck semi-rings of different parts of $\mathcal{ACVF}_{0,0}$. The precise statements and (some of) their proofs are explained in the following sections.

2. DEFINABLE SETS IN \mathcal{ACVF}

From now on, we will be concerned with the theory of algebraically closed valued fields, as defined below. We will state some key results, and provide some proofs from [12]. We follow [12] rather closely: all results and their proofs are from there, unless stated otherwise, though some of the more fundamental results there originate from [11].

The word “definable” will mean from now on: definable without parameters

signature of valued fields

Definition 2.0.1. (1) The *signature of valued fields* consists of the following data

- (a) Three sorts \mathbf{VF} , \mathbf{RV} and $\mathbf{\Gamma}$
- (b) The signature of rings on \mathbf{VF} , the signature of ordered abelian groups on $\mathbf{\Gamma}$, and the signature of rings on \mathbf{RV}
- (c) Function symbols $rv : \mathbf{VF} \rightarrow \mathbf{RV}$ and $v_{\mathbf{RV}} : \mathbf{RV} \rightarrow \mathbf{\Gamma} \cup \{\infty\}$

We will write $\mathbf{RV}^\times = \mathbf{RV} \setminus \{0\}$

theory of valued fields

(2) The *theory of valued fields*, \mathcal{VF} , is the (universal) theory in the above signature, axiomatised by:

- (a) $\mathbf{\Gamma}$ is an ordered abelian group, $(\mathbf{RV}^\times, \cdot)$ is an abelian group, $v_{\mathbf{RV}}(0) = \infty$, and $v_{\mathbf{RV}}$ restricts to a group homomorphism $v_{\mathbf{RV}} : (\mathbf{RV}^\times, \cdot) \rightarrow \mathbf{\Gamma}$.
- (b) \mathbf{VF} is a field, rv restricted to \mathbf{VF}^\times is a homomorphism of multiplicative groups, and $v := v_{\mathbf{RV}} \circ rv$ is a valuation, such that $v(x) > 0$ if and only if $rv(1+x) = 1$.
- (c) $+_{\mathbf{RV}}$ is a partially defined operation, so that $rv(x+y) = rv(x) + rv(y)$ whenever the latter is defined

algebraically closed fields

(3) The theory \mathcal{ACVF} of *algebraically closed fields* is the extension of \mathcal{VF} by the axioms that \mathbf{VF} is algebraically closed, rv and $v_{\mathbf{RV}}$ are surjective, and $\mathbf{\Gamma}$ is infinite.

For $\gamma \in \mathbf{\Gamma}$, we denote by \mathbf{RV}_γ the (γ -) definable subset $v_{\mathbf{RV}}(x) = \gamma$ in \mathbf{RV} .

Exercise 2.0.2. Show that the signature induces on \mathbf{RV}_0 the structure of the residue field, with 0 removed, and the rv restricts to the residue map from the invertible elements of the valuation ring to \mathbf{RV}_0 .

Hence, we have an exact sequence

$$1 \rightarrow \mathbb{k}^\times \rightarrow \mathbf{RV}^\times \xrightarrow{v_{\mathbf{RV}}} \mathbf{\Gamma} \quad (22)$$

(and the map on the right is surjective for models of \mathcal{ACVF}). We note that the sequence splits, since for models of \mathcal{ACVF} , the residue field is algebraically

closed, hence \mathbb{k}^\times is divisible. However, the splitting is not canonical (and no definable splitting exists, as we will see later).

Exercise 2.0.3. Let K be a valued field in the above signature, and let $s, t \in \mathbf{RV}(K)$. Show:

- (1) $s + t$ is well defined if $s \neq -t$
- (2) If $v_{\mathbf{RV}}(s) > v_{\mathbf{RV}}(t)$, then $s + t = t$
- (3) If $v_{\mathbf{RV}}(s) = v_{\mathbf{RV}}(t)$ and $s \neq -t$, then $v_{\mathbf{RV}}(s + t) = v_{\mathbf{RV}}(s)$

2.1. Elimination of quantifiers and some consequences.

Theorem 2.1.1 (Robinson, [22]). *The theory \mathcal{ACVF} eliminates quantifiers. Its completions $\mathcal{ACVF}_{p,q}$ are determined by specifying the characteristic p and residue characteristic q . It is the model completion of \mathcal{VF} .*

Proof. We will use 1.2.7. Thus, we are given algebraically closed valued fields L_1 and L_2 , with a common substructure L_0 . As in Remark 1.2.8, we assume that we are given an element $a \in L_1$, and we are looking for an element satisfying its quantifier-free type in L_2 , which we assume to be saturated. Furthermore, we note the following:

- (1) \mathcal{ACVF} implies that Γ is a divisible ordered abelian group, hence it admits quantifier-elimination. Therefore, we may assume that $\Gamma(L_0) = \Gamma(L_1) = \Gamma(L_2)$ (as ordered abelian groups).
- (2) Likewise, \mathcal{ACVF} implies that the residue field is algebraically closed, hence eliminates quantifiers. Thus, we may assume that $\mathbf{RV}_0(L_1) = \mathbf{RV}_0(L_2) = \mathbf{RV}_0(L_0)$. It follows (e.g., by the splitting mentioned above), that we may assume $\mathbf{RV}(L_1) = \mathbf{RV}(L_2) = \mathbf{RV}(L_0)$.
- (3) $\mathbf{VF}(L_0)$ is a field. Furthermore, by 1.2.21, we may pass to the henselisation of L_0 , and assume that L_0 is Henselian.

Thus we have $a \in \mathbf{VF}(L_1)$, with $\mathbf{VF}(L_0)$ Henselian (from now on we write L_i instead of $\mathbf{VF}(L_i)$). Assume first that a is algebraic over L_0 . Then, since L_2 is algebraically closed, we may find $b \in L_2$ with $L_0(a) = L_0(b)$ as fields. By 1.2.19, they are also isomorphic as valued fields.

Extending successively, we may assume that L_0 is algebraically closed. Let us now consider the collection of formulas

$$\Sigma = \{rv(x - c) = rv(a - c) : c \in \mathbf{VF}(L_0)\} \quad (23)$$

We claim that Σ determines the quantifier-free type of a over L_0 . Indeed, a general atomic formula has the form $\phi(rv(p_1(x)), \dots, rv(p_k(x)))$, with ϕ a formula over L_0 in the \mathbf{RV} sort, and p_i are polynomials over $\mathbf{VF}(L_0)$. Hence its truth value is determined once we know $rv(p_i(x))$. But L_0 is algebraically closed, and rv is a multiplicative homomorphism, so this reduces to linear factors.

We are thus reduced to realising Σ in L_2 . By saturation, it suffices to realise a finite subset, $rv(x - c_i) = rv(a - c_i) =: r_i$. By Exercise 2.0.3, if $x, y \in \mathbf{VF}$ are such that $v(x) > v(y)$, then $rv(y) = rv(y - x)$. Applying this remark to $x = a - c_i$ and $y = a - c_j$, it follows that it suffices to satisfy

those equations $rv(x - c_i) = r_i$ for which $v_{RV}(r_i)$ is maximal, say γ . Furthermore, if $r_i \neq r_j$, then their difference exists, as an element of RV_γ , and $r_i - r_j = rv(a - c_i) - rv(a - c_j) = rv(c_j - c_i)$. Thus, if $rv(x - c_i) = r_i$, we automatically have $rv(x - c_j) = r_j$. It follows that we may assume that all r_i are equal, say to r . Similar arguments show that $v(c_i - c_j) > v_{RV}(r)$ for all i, j , and therefore each such equation implies all the others. Thus, we are reduced to solving $rv(x - c) = r$, which can be done since rv is surjective.

This completes the proof of quantifier-elimination. The other two statements follow, since for any (consistent) choice of characteristics there is a prime model, and the valuation on any field extends to the algebraic closure (the prime models are \mathbb{Q}_q^a in the mixed characteristic case, and $K(t)^a$ with the t -adic valuation, where K is the prime field, in the equal characteristic case). \square

Corollary 2.1.2. *In $ACVF$:*

- (1) *The structure on Γ and on \mathbb{k} is the pure divisible ordered abelian group and algebraically closed field, respectively.*
- (2) *The (model theoretic) algebraic closure of a structure L_0 coincides with the field theoretic one on \mathbf{VF} . The definable closure coincides with the Henselisation. On Γ , both definable and algebraic closure are given by the divisible hull. On \mathbf{RV} , definable and algebraic closure are determined by the same on Γ and \mathbb{k} .*

Proof. (1) Follows directly from quantifier elimination.

- (2) We prove only for \mathbf{VF} . The algebraic closure is clear, since the field theoretic algebraic closure is a model. For the definable closure, assume a is properly algebraic over a Henselian L_0 . There is, then, a field automorphism of the algebraic closure over L_0 moving a . Since L_0 is Henselian, this automorphism preserves the valuation, and by quantifier elimination, it is then an automorphism of the full structure, so a is not definable. \square

Remark 2.1.3. The last corollary describes the definable and algebraic closure inside the given sorts. For model theoretic purposes, it is often essential to know the definable and algebraic closure including imaginaries. We will see below that $ACVF$ does not eliminate imaginaries, and will describe $ACVF^{eq}$. Thus, our description so far is incomplete. \square

Corollary 2.1.2 describes the 0-definable subsets of Γ and \mathbb{k} . What sets are definable if we allow parameters?

Definition 2.1.4. Let X be a definable set in a theory \mathcal{T} . X is *stably embedded* if any subset $Y \subseteq X^n$ definable with parameters can be defined with parameters from X .

If X is stably embedded, then its properties as a standalone structure (i.e., of the theory of X with the induced structure) coincide with its properties when viewed as a definable set in the theory \mathcal{T} .

Proposition 2.1.5. *In \mathcal{ACVF} , the definable sets Γ , \mathbf{RV} and \mathbb{k} are stably embedded*

Proof. Follows immediately from quantifier elimination: If \mathbf{Y} is a subset of \mathbf{RV}^n , definable with some parameters $\mathbf{a} \in \mathbf{VF}$, they can only appear in a formula via $\text{rv}(\mathbf{a})$. Hence we may replace \mathbf{a} by $\text{rv}(\mathbf{a})$. \square

2.2. C-minimality. Let \mathcal{T}_0 be a theory, and let \mathcal{T} be an expansion of \mathcal{T}_0 . We say that a collection \mathbf{X}_i of definable sets in \mathcal{T} *generates* \mathcal{T} if for any definable set \mathbf{Y} there is a map from a product of \mathbf{X}_i onto \mathbf{Y} . \mathcal{T} is said to be \mathcal{T}_0 -*minimal* if there is a generating collection \mathbf{X}_i such that any definable subset of each \mathbf{X}_i , even with parameters, is definable (with parameters) in \mathcal{T}_0 .

generates
 \mathcal{T}_0 -minimal

Example 2.2.1. A theory \mathcal{T} is *strongly minimal* if it is \mathcal{T}_0 -minimal, where \mathcal{T}_0 is the theory of equality (i.e., the theory of an infinite set with no additional structure). Hence, every (parametrically) definable subset of a generating sort is finite or co-finite. The theory \mathcal{ACF} of algebraically closed fields is strongly minimal (by quantifier elimination). \square

strongly minimal

Example 2.2.2. A theory is *o-minimal* if it is \mathcal{T}_0 -minimal, where \mathcal{T}_0 is the theory of dense linear orders. Hence, a generating sort carries a definable dense linear order, and each of its definable subsets is a finite union of intervals. The theory \mathcal{DOAG} of divisible ordered abelian groups, as well as of real-closed fields are examples of such theories. \square

o-minimal

Quantifier-elimination for \mathcal{ACVF} shows that any definable subset of \mathbf{VF} is a boolean combination of balls. Thus, it is minimal for the theory axiomatising the behaviour of these balls, which we now define.

Definition 2.2.3. The theory \mathcal{UM} of *ultra-metric spaces* is formulated in the language with two sorts \mathbf{VF} and $\Gamma_\infty = \Gamma \cup \{\infty\}$, a binary relation $<$ on Γ_∞ , and a function symbol $\mathbf{v} : \mathbf{VF}^2 \rightarrow \Gamma_\infty$. The axioms state:

ultra-metric spaces

- (1) $(\Gamma_\infty, <)$ is a dense linear order, with biggest element ∞ (but no smallest element)
- (2) If $\mathbf{v}(x, y) = \infty$ then $x = y$
- (3) For each $\gamma \in \Gamma_\infty$, the relation $\mathbf{v}(x, y) \geq \gamma$ is an equivalence relation on \mathbf{VF} (hence so is the relation given by $\mathbf{v}(x, y) > \gamma$, for $\gamma < \infty$)
- (4) Every class of $\mathbf{v}(x, y) \geq \gamma$ contains infinitely many classes of $\mathbf{v}(x, y) > \gamma$.

A theory \mathcal{T} is *C-minimal* if it is \mathcal{UM} -minimal.

C-minimal

Exercise 2.2.4. Show that \mathcal{UM} eliminates quantifiers, and is complete

Example 2.2.5. Quantifier elimination for \mathcal{ACVF} shows that it is C-minimal (with $\mathbf{v}(x, y) = \mathbf{v}_{\mathcal{ACVF}}(x - y)$). More generally, this is true for expansions of \mathcal{ACVF} that do not add new definable subsets of \mathbf{VF} (one variable), such as certain expansions by analytic functions ([15]). \square

As with \mathcal{ACVF} , the theory \mathcal{UM} does not eliminate imaginaries. We now describe some of the imaginary sorts (of course, these sorts will also occur

in any C-minimal theory). The classes of $\mathbf{v}(x, y) \geq \gamma$ are called *closed balls*, of radius γ . The collection of all closed balls forms an imaginary sort: it is the quotient $\mathbf{B}^c = \mathbf{VF} \times \Gamma / \sim$, where $(x_1, \gamma_1) \sim (x_2, \gamma_2)$ if $\gamma_1 = \gamma_2$, and $\mathbf{v}(x_1, x_2) \geq \gamma_1$. The projection to Γ thus descends to the quotient, and determines a map $r : \mathbf{B}^c \rightarrow \Gamma_\infty$, the radius. The fibre of this map over an element γ is the γ -definable set \mathbf{B}^c_γ of closed balls of radius γ (In particular, $\mathbf{B}^c_\infty = \mathbf{VF}$.)

open balls

The set \mathbf{B}^o of *open balls*, is likewise definable uniformly in the radius: it is $\mathbf{VF} \times \Gamma / \sim$, where this time \sim is defined via the sharp inequality. An open ball is contained in a unique closed ball of the same radius, so we obtain a definable closure map $c : \mathbf{B}^o \rightarrow \mathbf{B}^c$, preserving the radius. If $b \in \mathbf{B}^c$ is a closed ball, we again denote by \mathbf{B}^o_b the fibre of c over b , i.e., the b -definable set of open balls of radius $r(b)$ inside b .

Exercise 2.2.6. Show that if \mathcal{T} is C-minimal, then Γ is o-minimal and each fibre \mathbf{B}^o_b is strongly minimal.

We now fix an element $* \in \mathbf{VF}$; in \mathcal{ACVF} we will interpret $*$ as 0. The element determines a (definable) section $\Gamma_\infty \rightarrow \mathbf{B}^c$ of the radius map, namely, an element $\gamma \in \Gamma_\infty$ corresponds to the closed ball around $*$ of radius γ (i.e., the class of $*$ with respect to $\mathbf{v}(x, y) \geq \gamma$). We thus identify Γ_∞ with the set of closed balls containing $*$.

We denote by \mathbf{RV} the subset of \mathbf{B}^o consisting of balls b not containing $*$, with $c(b) \in \Gamma$ (i.e., with $* \in c(b) \setminus b$). The closure map thus restricts to a map $v_{\mathbf{RV}} = c : \mathbf{RV} \rightarrow \Gamma$. The fibre \mathbf{RV}_b of $v_{\mathbf{RV}}$ over a ball $b \in \Gamma$ is thus the fibre \mathbf{B}^o_b of c , with one element removed. We note that, as a result of this removal, \mathbf{RV} is a set of *disjoint* open balls.

Exercise 2.2.7. Verify that the description above coincides with the algebraic description that was given for \mathcal{ACVF}

Definition 2.2.8. The collection of 0-definable sets of the form \mathbf{RV}_b is known as **RES** (the *residue structure*)

RES
residue structure

Note that \mathbf{RV}_b is definable precisely if b is, which may happen even if \mathbf{RV}_b has no definable point.

strongly C-minimal

We say that \mathcal{T} is *strongly C-minimal* if it is C-minimal, and both **RES** and Γ are stably embedded. By 2.1.5 and 2.2.5, this holds in \mathcal{ACVF} .

Remark 2.2.9. The function \mathbf{v} can be viewed as a generalised (ultra-) metric (where, as usual, we invert the meaning of the order), and the terminology is follows this point of view. An alternative point of view is to view the structure as describing a particular kind of *directed tree*: the vertices of the tree are the closed balls \mathbf{B}^c , and there is a path from b_1 to b_2 if $b_2 \subseteq b_1$. The leaves of the tree the balls of radius ∞ , i.e., the elements of \mathbf{VF} . The tree comes with an addition structure of the “level” of a given vertex, i.e., its radius. The set Γ_∞ of levels can be identified with any complete branch through the tree. Such a branch is determined by its leaf, as with $*$ above. The set open balls inside a closed ball b is the set of maximal subtrees of

the subtree determined by b (i.e., the set of “connected components” of the subtree under b , with b removed). The whole tree can be viewed as the unique “open ball” of radius $-\infty$ \square

2.3. Imaginaries and definable power sets. We make a small digression to introduce definable power sets, which will simplify the discussion later. We work in an arbitrary theory \mathcal{T} .

If $Z \subseteq X \times Y$ is a definable set, and A is a \mathcal{T} -structure, we set $Z_a = \{x \in X : (x, a) \in Z\}$, an A -definable subset of X (which we call the fibre over a). Thus, each $a \in Y(A)$ determines an A -definable subset of X . We say that the pair (Y, Z) is an *definable power set* of X if for any A , any A -definable subset of X arises as Z_a for a *unique* $a \in Y(A)$.

definable power set

If an A -definable subset W of X has the form Z_a for a unique $a \in A$ (with Z possibly depending on W), we say that W can be *defined with a canonical parameter* (and a is then a canonical parameter for W). Clearly, if X admits a definable power set, then every one of its subsets is definable with a canonical parameter.

defined with a canonical parameter

Example 2.3.1. Let X be the universe in the theory of an infinite set with equality (i.e., the formula $x = x$). Does X admit a definable power set? Let b and c be two distinct elements, and let $W = \{b, c\}$. Assume that $W = Z_a$ for some Z and a unique a (note that a is a tuple with entries in $\{b, c\}$). a cannot be empty, since an arbitrary two element set is not definable over \emptyset . On the other hand, if a is not empty, the automorphism of the universe that exchanges b and c move a to some distinct d , but fixes W . Thus, $W = Z_d$, contradicting uniqueness. Thus, W cannot be defined with a canonical parameter, and consequently X does not admit a definable power set. \square

Example 2.3.2. Consider now \mathbb{A}^1 , given by the formula $x = x$ in the theory \mathcal{ACF} , let $Y = \mathbb{A}^2$, and let $Z = \{(x, c, d) : x^2 + cx + d = 0\} \subseteq X \times Y$. Then every subset $W = \{a, b\}$ is represented uniquely as $Z_{c,d}$, with $c = -(a + b)$ and $d = ab$. Thus, any set of size 2 can be defined with a canonical parameter (in fact, (Y, Z) is the family of all non-empty subsets of X of size at most 2). More generally, subsets of size n are coded by polynomials of degree n whose roots are simple (this is an algebraic condition on the coefficients).

Does X admit a definable power set? Such a definable power set would have to contain as definable subsets the families Y_n of all definable subsets of X of size n , for each $n \in \mathbb{N}$. However, it is intuitively clear (and not hard to prove using, e.g., Morley rank or Zariski dimension) that Y_n cannot be contained in X^{n-1} . It follows that no single definable set can classify even all finite subsets of X . \square

Exercise 2.3.3. Show that a theory \mathcal{T} admits elimination of imaginaries if and only if for every structure A , every A -definable set W can be defined with a canonical parameter

Though example 2.3.2 shows that definable power sets in \mathcal{ACF} do not exist, it also shows that the collection of all finite subsets of \mathbb{A}^1 is coded by a *system* of definable sets. This motivates the following definition.

ind-definable set

Definition 2.3.4. An *ind-definable set* in a theory \mathcal{T} is a system $\mathbf{X} = (\mathbf{X}_\alpha)$ of definable sets and definable maps between them, which is filtering: For any \mathbf{X}_α and \mathbf{X}_β in the system, there are maps $\mathbf{X}_\alpha \rightarrow \mathbf{X}_\gamma$ and $\mathbf{X}_\beta \rightarrow \mathbf{X}_\gamma$ in \mathbf{X} , into a common object \mathbf{X}_γ , and for any two maps $f, g : \mathbf{X}_\alpha \rightarrow \mathbf{X}_\beta$, there is a map $h : \mathbf{X}_\beta \rightarrow \mathbf{X}_\gamma$ with $h \circ f = h \circ g$.

A map from one ind-definable set \mathbf{X} to another \mathbf{Y} consists of a definable map $f_\alpha : \mathbf{X}_\alpha \rightarrow \mathbf{Y}_\beta$ from each \mathbf{X}_α into some \mathbf{Y}_β (depending on α), such the obvious diagrams commute; but two such system of maps (f_α) and (g_α) are identified if for every α there are maps s and t in the system \mathbf{Y} with $s \circ f_\alpha = t \circ g_\alpha$.

In particular, a chain of inclusions of definable sets is ind-definable. For example, the set of all polynomials, or of all polynomials with simple roots, can be viewed as an ind-definable set in \mathcal{ACF} .

We view a definable set as an ind-definable set represented by a constant sequence. It is easy to see that (finite) Cartesian products of ind-definable sets are again ind-definable. If A is a structure, and $\mathbf{X} = (\mathbf{X}_\alpha)$ is an ind-definable set, we define $\mathbf{X}(A)$ to be the direct limit $\mathbf{X}(A) = \varinjlim \mathbf{X}_\alpha(A)$. In particular, it makes sense to ask whether a definable power set exists as an ind-definable set, i.e., for a definable set \mathbf{X} , are there ind-definable sets \mathbf{Y} and $\mathbf{Z} \subseteq \mathbf{X} \times \mathbf{Y}$, such that every definable subset of \mathbf{X} has the form \mathbf{Z}_a for a unique $a \in \mathbf{Y}(A)$.

Exercise 2.3.5. Show that \mathcal{T} admits elimination of imaginaries if and only if every definable set has an ind-definable power set.

Exercise 2.3.6. Assume that (\mathbf{Y}, \mathbf{Z}) is an ind-definable power set of \mathbf{X} . Show that for any family $\mathbf{T} \subseteq \mathbf{X} \times \mathbf{W}$, with \mathbf{T} and \mathbf{W} ind-definable, there is a unique map $f : \mathbf{W} \rightarrow \mathbf{Y}$ with $\mathbf{T}_a = \mathbf{Z}_{f(a)}$ for structures A and $a \in \mathbf{W}(A)$.²

In particular, any two ind-definable power sets for a given definable set \mathbf{X} are canonically isomorphic.

From now on, we assume that \mathcal{T} admits elimination of imaginaries. For every definable set \mathbf{X} , we denote by $(\underline{\mathbf{P}}(\mathbf{X}), \in_{\mathbf{X}})$ the definable power set.

Similarly, a definable map $f : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{Z}$ can be viewed as a definable family of maps f_a from \mathbf{X} to \mathbf{Z} , one for each $a \in \mathbf{Y}(A)$. (\mathbf{Y}, f) is called an *internal Hom* from \mathbf{X} to \mathbf{Z} if any A -definable map has the form f_a for a unique $a \in \mathbf{Y}(A)$. Again, it makes sense to ask this for \mathbf{Y} ind-definable, and if such and ind definable set (\mathbf{Y}, f) exists, we denote it by $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Z})$, and the map f by ev (evaluation).

internal Hom

²The converse is also true: if (\mathbf{Y}, \mathbf{Z}) is a pair with this property, then it is an ind-definable power set. This is the standard definition of the “power object” in an arbitrary category

Exercise 2.3.7. Show that ind-definable internal $\underline{\mathbf{Hom}}$ s exist if and only if ind-definable power sets exist (for all definable sets). Show also that definable maps $f : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{Z}$ correspond bijectively to definable maps $\mathbf{Y} \rightarrow \underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Z})$.

From now on, we say simply “ $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$ exists” to mean that it exists as an ind-definable set.

Exercise 2.3.8. Show that if $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$ exists for each definable \mathbf{Y} , then it also exists for each ind-definable \mathbf{Y} .

The last few exercise simplify proving elimination of imaginaries in some cases:

Proposition 2.3.9. *Assume that the theory \mathcal{T} is generated by a collection of sorts \mathbf{X}_α (in the sense that every definable set is a subset of some product of sets \mathbf{X}_α), and at least one of them has two definable elements. Then \mathcal{T} eliminates imaginaries if and only if $\underline{\mathbf{Hom}}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$ exists for all α, β .*

Proof. One direction is Exercises 2.3.5 and 2.3.7. Assume that the internal $\underline{\mathbf{Hom}}$ sets in the statement exist. We first claim that $\underline{\mathbf{Hom}}(\mathbf{X}_\alpha, \mathbf{Y})$ exists for an arbitrary definable set \mathbf{Y} . If \mathbf{Y} is a Cartesian product of some \mathbf{X}_β s, then $\underline{\mathbf{Hom}}(\mathbf{X}_\alpha, \mathbf{Y}) = \prod_{\beta} \underline{\mathbf{Hom}}(\mathbf{X}_\alpha, \mathbf{X}_\beta)$. A general definable set \mathbf{Y} is a subset of such a product, so $\underline{\mathbf{Hom}}(\mathbf{X}_\alpha, \mathbf{Y})$ is a subset of the above given by a definable condition.

Next, we claim that $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$ exists for \mathbf{X} a product of \mathbf{X}_α , by induction on the length of the product. If $\mathbf{X} = \mathbf{X}_1 \times \dots \times \mathbf{X}_{n+1}$ (with \mathbf{X}_i basic sorts), then $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y}) = \underline{\mathbf{Hom}}(\mathbf{X}_1 \times \dots \times \mathbf{X}_n, \underline{\mathbf{Hom}}(\mathbf{X}_{n+1}, \mathbf{Y}))$, where the right hand side exists by induction (and Exercise 2.3.8).

In particular, the definable power set $\underline{\mathbf{P}}(\mathbf{X}) = \underline{\mathbf{Hom}}(\mathbf{X}, 2)$ exists for all Cartesian products of basic sorts \mathbf{X} (where 2 is the set consisting of two definable elements). If \mathbf{Y} is a definable subset of \mathbf{X} , the definable power set $\underline{\mathbf{P}}(\mathbf{Y})$ is clearly a definable subset of $\underline{\mathbf{P}}(\mathbf{X})$. By Exercise 2.3.5 again, we are done. \square

Example 2.3.10. Let $\mathcal{T}_A = \mathcal{DOAG}_A$ be the theory of divisible ordered abelian groups, with parameters for a non-trivial structure A . The \mathcal{T}_A eliminates imaginaries: by the proposition, we need to show that the family of maps from the (unique) sort Γ to itself is ind-definable. By quantifier elimination, any such map is piecewise of the form $f(x) = qx + t$, with q rational. Different rationals q and elements determine different functions. Hence, the collection of functions of this shape is coded by a disjoint union, indexed by \mathbb{Q} , of copies of Γ . A general function is determined by specifying a finite partition of Γ (i.e., a finite tuple of elements of Γ), and a function of the above form on each. Clearly, this is ind-definable. \square

We mention also a relation between definable power sets and stable embeddedness:

Exercise 2.3.11. Assume \mathcal{T} admits EI. Show that a definable set \mathbf{X} is stably embedded if and only if $\underline{\mathbf{P}}(\mathbf{X})$ is (isomorphic to) an ind-definable set in the reduct generated by \mathbf{X} .

2.4. Orthogonality.

strongly orthogonal

Definition 2.4.1. Definable sets \mathbf{X} and \mathbf{Y} are *strongly orthogonal* if any definable subset of $\mathbf{X}^k \times \mathbf{Y}^m$ is a finite union of definable sets of the form $\mathbf{Z} \times \mathbf{W}$, with $\mathbf{Z} \subseteq \mathbf{X}^k$ and $\mathbf{W} \subseteq \mathbf{Y}^m$. They are *orthogonal* if they are strongly orthogonal over the algebraic closure.

orthogonal

Thus, orthogonality is the strongest form of independence between definable sets: no non-trivial relations exist.

stable

We recall that a definable set \mathbf{X} is *stable* if there is no formula $\phi(x, y)$, even over parameters, that determines a linear order on an infinite set A of realisations of \mathbf{X} (we will only use this notion when \mathbf{X} is stably embedded). Any Cartesian product of stable sets is stable, and any strongly-minimal set is stable. In particular, the **RES** sorts in \mathcal{ACVF} are stable.

Proposition 2.4.2. *Assume \mathbf{X} and \mathbf{Y} are stably embedded definable sets in a theory \mathcal{T} , such that \mathbf{X} is stable, and \mathbf{Y} admits a definable order. Then \mathbf{X} and \mathbf{Y} are strongly orthogonal. In particular, in a strongly C-minimal theory \mathcal{T} , $\mathbf{\Gamma}$ and **RES** are strongly orthogonal.*

Proof. We first note that any definable map from \mathbf{X} to \mathbf{Y} is piecewise constant: if $f : \mathbf{X} \rightarrow \mathbf{Y}$ has infinite image, the formula $f(x_1) < f(x_2)$ orders an infinite set. The same is true for maps from \mathbf{X}^k to \mathbf{Y}^m .

Now, any subset $\mathbf{Z} \subseteq \mathbf{X}^k \times \mathbf{Y}^m$ determines a map $\mathbf{X}^k \rightarrow \underline{\mathbf{P}}(\mathbf{Y}^m)$ (mapping α to \mathbf{Z}_α). Since \mathbf{Y} is stably embedded, this map factors through some \mathbf{Y}^l , hence has finite image. Since \mathbf{Y} is ordered, each element of the image is definable, and so we may assume that the map is constant. This means that the fibre \mathbf{Z}_α does not depend on α in this part of the domain, i.e. that \mathbf{Z} is a product. \square

Other sets of balls (such as **RV**) are more complicated, namely, they have both strongly minimal and o-minimal aspects. Nevertheless, using their relations with these parts we may deduce some results about maps between them.

Proposition 2.4.3. *Any infinite set of balls of a fixed radius γ admits, over parameters, a surjective map onto a strongly minimal set. Hence any map from a subset of $\mathbf{\Gamma}$ to such a set is piecewise constant*

Proof. Let \mathbf{X} be the union of the balls. By quantifier elimination, \mathbf{X} contains a closed ball \mathbf{B} of radius greater than γ . The set of maximal open sub-balls of \mathbf{B} is strongly minimal, and each of the original balls is contained in at most one of them. The map that assigns to each such original ball the maximal open sub-ball of \mathbf{B} containing it is surjective.

The second statement follows directly from the first and orthogonality. \square

Corollary 2.4.4. *In a strongly C-minimal theory \mathcal{T} , the map $v_{\mathbf{RV}} : \mathbf{RV} \rightarrow \Gamma$ admits no definable section over an infinite set. More generally, any definable subset of \mathbf{RV}^n whose projection to Γ^n has finite fibres is finite.*

We note that this is true even with parameters, since adding parameters does not change the assumptions.

Proof. Assume there is a definable section $s : \mathbf{I} \rightarrow \mathbf{RV}$, where $\mathbf{I} \subseteq \Gamma$ is an infinite definable subset, which we may assume to be an interval (by o-minimality). By C-minimality, the union $\cup_{\gamma \in \mathbf{I}} s(\gamma)$ is a finite union of swiss cheeses, and by further reducing \mathbf{I} , we may assume it is one swiss cheese, a ball \mathbf{B} of radius γ , with a finite number of smaller balls removed. By passing to a slightly smaller ball, we may assume that \mathbf{B} is closed.

Consider the map s' on \mathbf{I} attaching to δ the open ball of radius γ inside \mathbf{B} that contains $s(\delta)$. By Prop. 2.4.3, s' has finite image. But this means that all $s(\delta)$, for all $\delta \in \mathbf{I}$, are contained in a finite union of balls properly contained in \mathbf{B} , contradicting the definition of \mathbf{B} .

The more general statement follows by induction (exercise). \square

Proposition 2.4.5. *Any infinite set of closed balls admits, over parameters, a surjective map onto an o-minimal set. Hence, any map from a stable set to a set of closed balls is piecewise constant*

Proof. If the radius map on the set of balls has infinite image, we are done. Otherwise, we may assume all balls have the same radius γ . As in the proof of 2.4.4, the union of all these balls contains a closed ball of radius $\delta > \gamma$, with a finite number of sub-balls removed. If c is any fixed point of this set, the function that sends a ball to its distance from c has infinite image. \square

3. GROTHENDIECK SEMI-RINGS IN \mathcal{ACVF}

We continue the study of definable sets in \mathcal{ACVF} , with the aim of proving some of the main results of [12]. The results themselves are explained in §3.4. We sketch the general structure

However, we start with an elementary formulation of those parts whose proofs we explain.

An application in the spirit of the classical p-adic integration is explained in the end of §3.3.

3.1. Surjectivity of the pullback from \mathbf{RV} . We now go back to the setting of valued fields: we assume that \mathcal{T} is a C-minimal theory expanding \mathcal{VF} . Our goal is to prove that up to “measure preserving” transformations, every subset of \mathbf{VF}^n (and more generally, of $\mathbf{VF}^n \times \mathbf{RV}^m$) is piecewise the pullback of some subset in \mathbf{RV} . This will imply that we may define a “measure” on definable subsets in \mathbf{VF} by pulling back a suitably defined measure from \mathbf{RV} (suitably defined means that the measure should be the same on subsets of \mathbf{RV} that pull back to the same class of subsets of \mathbf{VF}). In terms of

Grothendieck semirings, this will imply that the map on the level of semirings is surjective. However, at this point, these semirings are not yet defined.

The plan above, and the definition of the Grothendieck semiring of \mathbf{VF} , depend upon the right notion of “measure preserving” maps. Since the statement becomes stronger with a smaller class of maps, the following notion of “measure preserving” is reasonable.

admissible map

Definition 3.1.1. Let n be fixed. An *admissible map* is a definable map which is a composition of maps of the following forms

(1)

$$(\bar{x}, \bar{y}) \mapsto (x_1, \dots, x_i + a(x_1, \dots, x_{i-1}, \bar{y}), x_{i+1}, \dots, x_n, \bar{y}) \quad (24)$$

where \bar{x} are \mathbf{VF} variables, \bar{y} are \mathbf{RV} -variables, and a is definable

(2) $(\bar{x}, \bar{y}) \mapsto (\bar{x}, \bar{y}, \text{rv}(x_i))$

As mentioned above, we would like to prove that every definable subset of \mathbf{VF}^n is, up to an admissible transformation, piecewise a pullback from \mathbf{RV} . To have any hope for such a statement to be true, we need to have “sufficiently many” admissible transformations: for example, we need to be able to shift a ball sufficiently close to 0. This is equivalent to having a definable point (centre) in the ball. We note that in general, if B is a definable ball, there need not be a definable element (over 0) inside B . Hence, we need an additional assumption, expressed through the following definition.

centred closed balls

Definition 3.1.2. A \mathcal{C} -minimal theory \mathcal{T} of valued fields is said to have *centred closed balls* if any definable closed ball in \mathcal{T} has a definable centre³

We may now formulate the main result.

Proposition 3.1.3. *Assume that \mathcal{T} is a \mathcal{C} -minimal theory of valued fields, such that for any \mathcal{T} -structure \mathcal{A} , $\mathcal{T}_{\mathcal{A}}$ has centred closed balls.*

Let \mathbf{X} be a definable subset of $\mathbf{VF}^n \times \mathbf{RV}^l$. Then there is a definable partition of \mathbf{X} into finitely many pieces \mathbf{Z} , such that for each \mathbf{Z} we have $\mathbf{TZ} = (\text{rv}, 1)^{-1}(\mathbf{H})$ for some definable \mathbf{H} in \mathbf{RV} and some admissible transformation \mathbf{T} .

If \mathbf{Z} is bounded, then so is \mathbf{H} .

For subsets of \mathbf{RV}^l , by *bounded* we mean bounded from below, i.e., $v_{\mathbf{RV}}(r) \geq \gamma$ for some γ . We will see below that the assumptions of the proposition hold in the case of $\mathcal{ACVF}_{0,0}$.

Baring in mind that \mathbf{RV} is a set of balls, this statement can be thought of as the analogue to the description of definable sets that occurs in the proof of Theorem 1.1.4 in Denef’s theory (the description here is actually simpler, since the root predicates do not occur). As in that proof, the result follows by induction and compactness from the case of a single coordinate, which we now formulate (this will be the analogue of 1.3.8)

³This is slightly stronger than the definition in [12], but coincides with it in residue characteristic 0

Lemma 3.1.4. *Assume \mathcal{T} as in Prop. 3.1.3. Then any definable subset of \mathbf{VF} is a finite disjoint union of definable set \mathbf{Z} with $\mathbf{TZ} = (\text{rv}, 1)^{-1}(\mathbf{H})$ with \mathbf{T} admissible, and \mathbf{H} definable in \mathbf{RV} . If \mathbf{Z} is bounded, then so is \mathbf{H} .*

We note that admissible maps in one \mathbf{VF} variable are particularly simple, so the lemma says: \mathbf{Z} is piecewise of the form $\mathfrak{a} + \text{rv}^{-1}(\mathbf{H})$ for $\mathfrak{a} \in \mathbf{VF}$ and \mathbf{H} definable in \mathbf{RV} .

Exercise 3.1.5. Deduce Proposition 3.1.3 from Lemma 3.1.4

Proof of 3.1.4. We consider several cases for \mathbf{X}

A closed ball. After translation, $\mathbf{X} = \mathfrak{v}^{-1}([\gamma, \infty])$ (the translation exists by the centred closed balls assumption)

An open ball. After translation, the closure of \mathbf{X} is around 0. If \mathbf{X} itself is around 0, it is the inverse image of a half-open interval, otherwise it is (the inverse image of) a point in \mathbf{RV}

A ball with a closed ball removed. Exercise

A closed ball with a finite number of open balls of the same radius removed. Exercise

A union of m of the above. By induction on m . Let \mathbf{E} be the smallest ball containing \mathbf{X} . Since $m > 1$, \mathbf{E} is closed, and we may assume it contains 0. By minimality of \mathbf{E} , \mathbf{X} is not contained in a maximal open sub-ball, and we may assume that it does not intersect any open sub-ball of \mathbf{E} around 0 (otherwise, the part that intersects is definable and divides \mathbf{X} into pieces with smaller m , and we are done by induction).

Hence, $\text{rv}(\mathbf{X})$ has more than one point. The fibres \mathbf{X}_b of rv admit, by induction, admissible maps \mathbf{T}_b and definable \mathbf{H}_b as stated, which can be chosen uniformly. Hence $\mathbf{T}(x) = (\mathbf{T}_{\text{rv}(x)}(x), \text{rv}(x))$ solves the problem.

The general case. We may assume that \mathbf{X} is a finite union of balls of the same radius, with some balls removed (exercise). Let β be the distance between the closest holes of \mathbf{X} . Let \mathbf{C} be the union of closed balls of radius β around the holes of \mathbf{X} . Then \mathbf{X} is the disjoint union of $\mathbf{X} \setminus \mathbf{C}$ and $\mathbf{C} \cap \mathbf{X}$. The first has fewer holes, so follows by induction, hence we may assume that $\mathbf{C} \cap \mathbf{X} = \mathbf{X}$. Let \mathbf{D} be the union of maximal open sub-balls that contain the holes of \mathbf{X} . The \mathbf{X} is a disjoint union of $\mathbf{C} \setminus \mathbf{D}$ and \mathbf{D} . Each of these reduces to the previous special cases. \square

3.2. Centred closed balls. We now show that the assumption of Proposition 3.1.3, namely, that \mathcal{T}_A has centred closed balls (def. 3.1.2) holds for $\mathcal{T} = \mathcal{ACVF}_{0,0}$. The assumption on the characteristic plays a role via the following observation.

Lemma 3.2.1. *Assume that a ball in $\mathcal{ACVF}_{0,0}$ contains a finite definable set. Then it contains a definable point.*

Proof. We may assume that the finite set $A = \{a_1, \dots, a_n\}$ is a single orbit under the Galois group. Also, we may assume the ball is closed, the smallest closed ball containing A .

Let $a = \frac{a_1 + \dots + a_n}{n}$. Then a is definable, and

$$|a - a_1| = \left| \frac{a_1 - a_1}{n} + \dots + \frac{a_n - a_1}{n} \right| \leq \max_i |a_i - a_1| \quad (25)$$

Each element in the maximum on the right is at most the radius of the ball, hence so is the left hand side. Thus, a is in the ball. \square

It follows from this lemma that if A is generated by its **VF** part, then every closed ball has as centre, since in this case the algebraic closure of A is a model. However, not every structure is generated by its **VF** part. If A is a model, and we add an element b to $\Gamma(A)$ or to $\mathbf{RV}(A)$, there will be new b -definable balls, which we do not yet know to contain a centre. However, we will show that any such ball contains an A -definable ball, and the result will follow.

To state the following lemma, we need an additional definition. Recall the p -adic fields are complete with respect to their absolute value. This is not a first order property, but we have the following definable analogue.

definably complete

Definition 3.2.2. A C -minimal theory is *definably complete* if any definable nested family of balls has a non-empty intersection

A family of balls is nested if no two are disjoint. Note that the intersection is a definable set, so this is indeed a property of the theory. A model of the theory may have the property that *any* nested family of balls has a non-empty intersection. In this case, the theory is clearly definably complete. This is the situation in \mathcal{ACVF} : such models are called *spherically complete*.

spherically complete

We remark also that the condition extends to type-definable families of nested balls, since each such family is contained in a definable one.

Lemma 3.2.3. *Let \mathcal{T} be a definably complete C -minimal theory, and let Γ_0 be a set of elements from Γ . Then any non-empty Γ_0 -definable ball contains a non-empty definable sub-ball.*

Proof. We may assume that Γ_0 is finite, and by induction to consists of one element γ_0 . Let $\mathbf{X} \subseteq \Gamma$ be the type of γ_0 . For every $\gamma \in \mathbf{X}$, we are given a ball \mathbf{B}_γ defined over γ . We wish to show that the family \mathbf{B}_γ (indexed by $\gamma \in \mathbf{X}$) is nested. This will complete the proof, since the intersection will be a definable sub-ball, which is non-empty by definable completeness.

The radius $r(\gamma)$ of the ball \mathbf{B}_γ is a definable function from \mathbf{X} to Γ , hence monotone, and we will assume for simplicity of notation that it is the identity, i.e., \mathbf{B}_γ is of radius γ . Thus, we wish to show that if $\gamma_1 < \gamma_2$, then $\mathbf{B}_{\gamma_1} \subseteq \mathbf{B}_{\gamma_2}$.

For $\gamma_1 < \gamma_2$ in \mathbf{X} , let $d(\gamma_1, \gamma_2)$ be the distance between \mathbf{B}_{γ_1} and \mathbf{B}_{γ_2} . Suppose that for some γ_2 , for all sufficiently close $\gamma_1 < \gamma_2$ we have $d(\gamma_1, \gamma_2) \geq \gamma_2$. Then for some $\gamma > \gamma_2$ sufficiently close to γ_2 we also have $d(\gamma_1, \gamma_2) > \gamma$ (since $d(\gamma_1, \gamma_2)$ is defined over γ_1, γ_2). In other words, either $\mathbf{B}_{\gamma_1} \subseteq \mathbf{B}_{\gamma_2}$, or the distance between them is at least γ .

Let \mathbf{B}'_{γ_i} be the open ball of radius γ around \mathbf{B}_{γ_i} . Then we have seen that either $\mathbf{B}'_{\gamma_1} \subset \mathbf{B}'_{\gamma_2}$ for $\gamma_1 < \gamma_2 < \gamma$ (sufficiently close to γ), or the balls \mathbf{B}'_{γ} are all distinct. In the first case we are done, since γ and γ_i all have the same type. In the second case, we have an injective definable map from an infinite subset of Γ to a set of balls of the same radius. This is excluded by Prop. 2.4.3. \square

The last lemma allows us to reduce to the case where the map $v : \mathbf{VF}(A) \rightarrow \Gamma(A)$ is surjective. Next, we deal with \mathbf{RV} .

Lemma 3.2.4. *Let \mathcal{T} be a definably complete C -minimal theory, and let A_0 be a set of elements from \mathbf{RV} . Then any non-empty A_0 -definable ball contains a non-empty definable closed sub-ball.*

Proof. As before, we may assume that A_0 consists of one element, a . By Lemma 3.2.3, we may assume that $\gamma = v_{\mathbf{RV}}(a)$ is definable. Hence, we have a definable family of closed balls \mathbf{B}_x , with x ranging over the strongly minimal set \mathbf{RV}_γ . By Prop. 2.4.5, this family is constant, i.e., \mathbf{B}_a itself is definable. \square

To summarise, we have the following.

Corollary 3.2.5. *Let $\mathcal{T} = \mathcal{ACVF}_{0,0}$. For any \mathcal{T} -structure A , the theory \mathcal{T}_A admits centred closed balls. Hence, in $\mathcal{ACVF}_{0,0}$, any definable set is, up to admissible transformations, a pull back of a definable set in $\mathbf{RV} \cup \Gamma$.*

Proof. Let A_0 be the substructure of A generated by $\mathbf{VF}(A)$. If \mathbf{B} is a non-empty closed ball defined over A , by Lemmas 3.2.3 and 3.2.4 it contains a non-empty closed ball \mathbf{B}' defined over A_0 . Since the algebraic closure of A_0 is a model, \mathbf{B}' contains a point there. By Lemma 3.2.1, it also has a point in A_0 .

The second statement follows from Prop. 3.1.3. \square

We also note the following corollary regarding maps into \mathbf{VF}

Corollary 3.2.6. *Any map from \mathbf{RV}^n (and hence from Γ^n) to a set of disjoint closed balls is piecewise constant*

This applies, in particular, to balls of radius 0, i.e., elements of \mathbf{VF} .

Proof. The closed ball given by Lemma 3.2.4 is contained in any ball in the image (after a finite partition of the domain). Since all balls in the image are disjoint, the map must be constant. \square

3.3. Surjectivity of the map to the Grothendieck group of \mathbf{RV} . Let R be a ring. A *definable R -module* in a theory \mathcal{T} is a definable abelian group \mathbf{A} in \mathcal{T} , together with a map of rings from R to the ring of definable endomorphisms of \mathbf{A} . definable R -module

Proposition 3.3.1. *Let R be a ring, and assume $\pi : \mathbf{B} \rightarrow \mathbf{C}$ is a surjective map of definable R -modules in \mathcal{T} , with kernel \mathbf{A} . Assume:*

- (1) \mathbf{A} and \mathbf{C} are stably embedded and orthogonal

- (2) Every definable subgroup of \mathbf{A}^n is defined by R -linear equations.
(3) If the restriction of π to some definable $\mathbf{P} \subseteq \mathbf{B}^n$ has finite fibres, then \mathbf{P} is finite.

Then every definable subset of \mathbf{B}^n is a finite union of sets of the form $\pi^{-1}(\mathbf{W}) \cap r^{-1}(\mathbf{Y} + \mathbf{b})$, where $\mathbf{W} \subseteq \mathbf{C}$, r is a matrix over R , $\mathbf{Y} \subseteq \mathbf{A}^k$, and $\mathbf{b} \in \mathbf{B}^k$.

We note that the assumptions above hold for $\mathbf{B} = \mathbf{R}\mathbf{V}$ and $\pi = v_{\mathbf{R}\mathbf{V}}$ (and $R = \mathbb{Z}$) in \mathcal{ACVF}

Proof. Replacing R with $M_n(R)$, we may assume $n = 1$. Let $\mathbf{Z} \subseteq \mathbf{B}$. The assignments $c \mapsto \mathbf{Z}_c$ determines a map $\mathbf{C} \rightarrow \underline{\mathbf{P}}(\mathbf{B})$. The addition induces an action of \mathbf{B} on $\underline{\mathbf{P}}(\mathbf{B})$ (by translation), so we obtain a map $\tilde{s} : \mathbf{C} \rightarrow \underline{\mathbf{P}}(\mathbf{B})/\mathbf{B}$. Since every element \mathbf{Z}_c is a subset of a coset, this map factors through $\underline{\mathbf{P}}(\mathbf{A})/\mathbf{A}$. But \mathbf{A} is stably embedded, and orthogonal to \mathbf{C} , so this map is piecewise constant, and we may assume it is constant. Let $\mathbf{A}_0 \subseteq \mathbf{A}$ represent this constant element.

Let \mathbf{S} be the stabiliser of \mathbf{A}_0 . Since \mathbf{S} fixes all the fibres \mathbf{Z}_c (as sets), it also fixes \mathbf{Z} as a set over \mathbf{C} , so \mathbf{S} is definable. The definable set \mathbf{Z}/\mathbf{S} admits a well-defined map $\bar{\pi}$ to \mathbf{C} . If z_1 and z_2 are two elements in the same fibre \mathbf{Z}_c , then they differ by an element of \mathbf{S} , hence we have a well defined section $s : \pi(\mathbf{Z}) \rightarrow \mathbf{Z}/\mathbf{S}$. It follows that if $r \in R$ is 0 on \mathbf{S} , we have a well defined map $s_r = rs : \pi(\mathbf{Z}) \rightarrow r\mathbf{Z}$, so that $r\mathbf{Z}_c = s_r(c) + r\mathbf{A}_0$.

Fix r as above. Then for $d \in \mathbf{C}$ in the kernel of r , we have $\pi s_r(d + c) = r(d + c) = rc = \pi s_r(c)$, hence $s_r(d + c) - s_r(c) \in \mathbf{A}$. Again by orthogonality, this is a piecewise constant function of d, c . It follows that the image of s_r projects with finite fibres to \mathbf{C} . By the third assumption, it is finite. \square

Corollary 3.3.2. *Every definable subset of $\mathbf{R}\mathbf{V}^n$ is a finite disjoint union of pieces \mathbf{X} , each of the form $dx = T(\mathbf{Y} \times \mathbf{Z})$, where $\mathbf{Z} = v_{\mathbf{R}\mathbf{V}}^{-1}(\mathbf{W})$ with $\mathbf{W} \subseteq \Gamma^k$, $v_{\mathbf{R}\mathbf{V}}(\mathbf{Y})$ consists of one point (i.e., is a translate of a constructible set in the residue field), and T is an invertible \mathbb{Z} -linear transformation.*

Combining with 3.2.5, we obtains one direction of the integration results:

Corollary 3.3.3. *Any definable subset of $\mathbf{V}\mathbf{F}^n \times \mathbf{R}\mathbf{V}^l$ can be partitioned into finitely many pieces of the form $T(\mathbf{X} \times \Delta)$, where T is an admissible transformation, \mathbf{X} is a definable subset in $\mathbf{R}\mathbf{E}\mathbf{S}$, and Δ is a definable subset in Γ .*

The following corollary is a uniform version of the integration type results that were discussed in §1. As explained in the proof, we obtain a conclusion about discretely valued fields in mixed characteristic, despite the fact that we were working in $\mathcal{ACVF}_{0,0}$. This is essentially due to the quantifier free nature of the statement. It is analogous to the original results 1.1.1 by Igusa, rather than the more general results by Denef. Results of the latter type can also be dealt with, but with additional effort (c.f. [12, §12]).

Corollary 3.3.4. *Let $f = (f_1(x_1, \dots, x_n), \dots, f_k(x_1, \dots, x_n))$ be polynomials over \mathbb{Z} . Then there are finitely many generalised varieties \mathbf{X}_i and polyhedra Δ_i*

such that for sufficiently large prime p , for any local field L of residue characteristic p , we have

$$\int_{\mathcal{O}^n} |f|^s = \sum_i q^{-r\gamma(i)} (q-1)^{n_i} \#X_i(L) \text{ev}_L(\Delta_i) \quad (26)$$

Where r is the ramification degree of L , q is the size of the residue field, $\gamma(i)$ and n_i are non-negative rational and integer (resp.), and

$$\text{ev}_L(\Delta) = \sum_{(x,y) \in \Delta(L)} q^{-r((\sum y) + s \cdot x)} \quad (27)$$

Proof. Consider the definable subset \mathbf{Z} of $\mathbf{VF}^n \times \Gamma^k$ given by the graph of $v \circ f$. As usual,

$$\int_{\mathcal{O}^n} |f|^s = \sum_{\alpha \in \Gamma^k} q^{r(s \cdot \alpha)} \mu(\mathbf{Z}_\alpha) \quad (28)$$

On the other hand, by the previous corollary, \mathbb{Z} is, up to admissible transformations, piecewise of the form $rv^{-1}(\mathbf{X}_i) \times v^{-1}(\Delta_i)$ for some generalised varieties \mathbf{X}_i and polyhedra Δ_i . This is true in $\mathcal{ACVF}_{0,0}$. However, we note that all sets involved are quantifier-free, and therefore this also holds in substructures. Furthermore, the statement depends only on finitely many axioms of $\mathcal{ACVF}_{0,0}$, hence also holds in prime (residue) characteristic for sufficiently large prime.

It follows by translation invariance, additivity and Fubini, that $\mu(\mathbf{Z}_\alpha) = \sum_i \mu(rv^{-1}(\mathbf{X}_i)) \mu(v^{-1}(\Delta_{i_\alpha}))$, so it remains to compute each term separately.

For \mathbf{X} a generalised variety, we may assume that \mathbf{X} is contained in one fibre of v_{RV} , the fibre over $\bar{\gamma}$. Then each point of $\mathbf{X}(L)$ represents a product of open balls, of radii γ_j , and the measure of such a product is $q^{-r \sum_j \gamma_j}$. Hence $\mu(rv^{-1}(\mathbf{X})) = q^{-r\gamma} \# \mathbf{X}(L)$, where $\gamma = \sum_j \gamma_j$.

On the other hand, given $\bar{\gamma} \in \Gamma^n$, the set $v^{-1}(\bar{\gamma})$ is a product of n “thin annuli”, of radii γ_i , so its measure is $\prod_j (q-1)q^{-r\gamma_j} = (q-1)^n q^{-r\gamma}$, with $\gamma = \sum_j \gamma_j$. Therefore, for a polyhedron Δ we have $\mu(v^{-1}(\Delta_\alpha)) = (q-1)^n \sum_{(a,\bar{\gamma}) \in \Delta} q^{-r\gamma}$, where $\Delta_\alpha = \{\bar{\gamma} \in \Gamma^n : (a,\bar{\gamma}) \in \Delta\}$ and, as before, $\gamma = \sum_j \gamma_j$. Combining the computations, we get the result. \square

3.4. Grothendieck semi-rings, and main results. We will now state some of the main results of [12]. Recall that we are interested in assigning, to each definable subset of $\mathbf{VF}^* \times \mathbf{RV}^*$ a universal “measure”, taking values in (variants of) the Grothendieck rings of generalised varieties and of Γ . This measure should be invariant under “measure preserving” transformations, additive with respect to disjoint unions, and multiplicative on Cartesian products. In other words, the domain of the measure is also a Grothendieck ring, the Grothendieck of definable sets in \mathcal{ACVF} and certain definable maps.

We make two more general remarks: First, the Grothendieck semi-groups and semi-rings (with no additive inverse) carry substantially more information than the corresponding rings, since cancellation induces strong relations. For example, if a is a definable element of the valuation ring, then

the valuation ring \mathcal{O} and the ideal generated by \mathfrak{a} are isomorphic, so the class of the annulus $\mathcal{O} \setminus \mathfrak{a}\mathcal{O}$ is 0 in the Grothendieck ring of \mathcal{ACVF} (with all definable bijections). This need not be the case in the semi-ring.

The other remark is that there are actually infinitely many measures, one for each dimension. For example, the measure of the valuation ring as a subset of \mathbb{A}^1 should not be the same as its image under a linear embedding in \mathbb{A}^2 .

Hence, we expect the motivic measure to be a map from a certain graded Grothendieck semi-ring of definable sets in \mathcal{ACVF} to a graded semi-ring of definable sets in \mathbf{RES} and $\mathbf{\Gamma}$. This is indeed the result, but it is convenient to pass through the Grothendieck semi-ring of \mathbf{RV} . There several variants of the main result, depending on the kind of structure one wishes to preserve. We describe in some detail one where no measure is actually preserved, i.e., we compute the Grothendieck semi-ring of \mathbf{VF} , with all definable maps. Other variant are similar, and mostly follow from this case, but involve a careful definition of measure preserving maps.

We begin by defining dimensions, which determine the grading. We will say that a definable set \mathbf{X} is *quasi-finite* over a definable set \mathbf{Y} if there is a definable map with parameters from \mathbf{X} to \mathbf{Y} , whose fibres are finite (hence, the dimension of \mathbf{X} should be at most that of \mathbf{Y}).

Definition 3.4.1. Let \mathbf{X} be a definable set in \mathcal{ACVF} .

- (1) The *VF-dimension* of \mathbf{X} is the smallest n for which \mathbf{X} is quasi-finite over $\mathbf{VF}^n \times \mathbf{RV}^* \times \mathbf{\Gamma}^*$.
- (2) The *RV-dimension* of \mathbf{X} is the smallest n for which \mathbf{X} is quasi-finite over $(\mathbf{RV} \cup \mathbf{\Gamma})^n$ (if such an n exists)

Only finite subsets of \mathbf{VF}^n have an *RV-dimension*. On the other hand, note that the *VF-dimension* of a subset of \mathbf{RV} (or $\mathbf{\Gamma}$) is 0. More generally, it is an exercise that \mathbf{X} has *VF-dimension* at most n if and only if there is a definable map $f : \mathbf{X} \rightarrow \mathbf{VF}^n$ (with parameters), whose fibres are in \mathbf{RV} . It is possible to show that the *RV-dimension* of the fibres in this case is essentially independent of f .

Definition 3.4.2. (1) $\mathbf{VF}[n, \cdot]$ is the category of definable sets of *VF-dimension* at most n , with morphisms all definable maps.

(2) $\mathbf{VF}[n]$ is the full subcategory of $\mathbf{VF}[n, \cdot]$ of objects for which the *RV-dimension* of the fibres of a map to \mathbf{VF}^n is 0.

(3) $\mathbf{RV}[n, \cdot]$ is the category whose objects are pairs (\mathbf{X}, f) , with $\mathbf{X} \subseteq \mathbf{RV}^*$ and $f : \mathbf{X} \rightarrow \mathbf{RV}^n$ is a definable map. A morphism from (\mathbf{X}, f) to (\mathbf{Y}, g) is a map that determines a correspondence on the base, i.e., a map $h : \mathbf{X} \rightarrow \mathbf{Y}$ such that the first projection of $\{(f(x), g(h(x))) : x \in \mathbf{X}\}$ is finite-to-one.

(4) $\mathbf{RV}[n]$ is the full subcategory of $\mathbf{RV}[n, \cdot]$ consisting objects (\mathbf{X}, f) where the fibres of f are finite.

(5) $\mathbf{RES}[n]$ is the full subcategory of $\mathbf{RV}[n]$ consisting of objects (\mathbf{X}, f)

quasi-finite

VF-dimension

RV-dimension

 $\mathbf{VF}[n, \cdot]$ $\mathbf{VF}[n]$ $\mathbf{RV}[n, \cdot]$ $\mathbf{RV}[n]$ $\mathbf{RES}[n]$

with $\mathbf{X} \subseteq \mathbf{RES}$.

- (6) $\Gamma[n]$ is the category whose objects are definable subsets of Γ^n , and whose morphisms are piecewise of the form $x \mapsto Bx + c$, where B is an integral matrix, and c is an element of the group of values of K . $\Gamma[n]$

We note that elements (and subsets) of \mathbf{RV} have a “dual nature” in terms of dimension: when pulling back a subset of \mathbf{RV} , one obtains a set of positive \mathbf{VF} -dimension. The \mathbf{RV} -dimension is designed to reflect this fact. On the other hand, the \mathbf{VF} -dimension of \mathbf{RV} itself is 0. Thus, an object (\mathbf{X}, f) of $\mathbf{RV}[n, \cdot]$ should be thought of as the n -dimensional subset $f(\mathbf{X})$, with some 0-dimensional fibres attached. A morphism in $\mathbf{RV}[n, \cdot]$ is thus viewed as a correspondence on $f(\mathbf{X})$, together with a matching lifting to the fibres.

We also let \mathbf{RV}_0 be the union of \mathbf{RV} with a formal element 0, and define $rv(0) = 0$, $v_{\mathbf{RV}}(0) = \infty \in \Gamma_\infty$. We define $\mathbf{RV}_0[n, \cdot]$ and $\mathbf{RV}_0[n]$ analogously to the case of \mathbf{RV} . Then an object (\mathbf{X}, f) of $\mathbf{RV}_0[n, \cdot]$ determines an object of $\mathbf{RV}[m, \cdot]$, where m is the number of non-zero coordinates of f (by erasing the 0 coordinates). This clearly determines an equivalence of $\mathbf{RV}_0[n, \cdot]$ with $\bigoplus_{m < \leq n} \mathbf{RV}[m, \cdot]$ (and likewise for $\mathbf{RV}_0[n]$).

We have seen in Prop. 3.2.5 how to obtain definable sets in \mathcal{ACVF} by pulling back from \mathbf{RV} . This can be reformulated as follows: For an object (\mathbf{X}, f) of $\mathbf{RV}_0[n, \cdot]$, let

$$\mathbb{L}(\mathbf{X}, f) = \mathbf{VF}^n \times_{\mathbf{RV}_0^n} \mathbf{X} = \{(y, x) \in \mathbf{VF}^n \times \mathbf{X} : rv(y) = f(x)\} \quad (29)$$

This is clearly an object of $\mathbf{VF}[n, \cdot]$. Note that $\mathbb{L}(\mathbf{X}, f) \in \mathbf{VF}[n]$ if and only if $(\mathbf{X}, f) \in \mathbf{RV}_0[n]$.

Exercise 3.4.3. Show that any object in $\mathbf{VF}[n, \cdot]$ is isomorphic, via an admissible transformation, to an object of the form $\mathbb{L}(\mathbf{X}, f)$.

The definition of morphisms in $\mathbf{RV}[n, \cdot]$ is motivated by the following observation.

Lemma 3.4.4. *Assume that (\mathbf{X}, f) and (\mathbf{Y}, g) are objects of $\mathbf{RV}[n, \cdot]$, and $h : \mathbf{X} \rightarrow \mathbf{Y}$ is a bijection that lifts to a bijection $H : \mathbb{L}(\mathbf{X}, f) \rightarrow \mathbb{L}(\mathbf{Y}, g)$. Then h is a morphism of $\mathbf{RV}[n, \cdot]$*

Proof. We need to show that the first projection from $\{(f(x), g(h(x))) : x \in \mathbf{X}\}$ has finite fibres, i.e., that for all $x \in \mathbf{X}$, $g(h(x))$ is algebraic over $f(x)$. Since H is a lift of h , we have $H(v, x) = (u(v, x), h(x))$ for some definable $u : \mathbb{L}(\mathbf{X}, f) \rightarrow \mathbf{VF}^n$, and $g(h(x)) = rv(u(v, x))$. By 3.2.6, for each $v \in \mathbf{VF}^n$, $u(v, x)$ is algebraic over v . So $g(h(x))$ is algebraic over (v, x) for all v satisfying $rv(v) = f(x)$. Hence, it is algebraic over $f(x)$. \square

It turns out that the converse of this lemma also holds: An isomorphism in $\mathbf{RV}[n, \cdot]$ can be lifted to a definable bijection in \mathbf{VF} . This is proved in [12, Prop. 6.1].

It follows that \mathbb{L} determines a map of semigroups $\mathbb{L} : K_+(\mathbf{RV}_0[n]) \rightarrow K_+(\mathbf{VF}[n])$. The semigroups $K_+(\mathbf{VF}[n])$ define an increasing filtration of

$K_+(\mathbf{VF})$, the Grothendieck semi-ring of all definable sets in \mathcal{ACVF} and bijections. Likewise, we have embeddings of $K_+(\mathbf{RV}_0[n]) \rightarrow K_+(\mathbf{RV}_0[n+1])$, obtained by adding an additional 0 coordinate, and the union $K_+(\mathbf{RV}[*]) = \varinjlim_n K_+(\mathbf{RV}_0[n])$ forms a filtered semi-ring, with the obvious product. Since \mathbb{L} clearly commutes with the product, we obtain from the above discussion and from 3.2.5 the following result.

Proposition 3.4.5. *The operation \mathbb{L} induces a surjective homomorphism of filtered semirings, $\mathbb{L} : K_+(\mathbf{RV}[*]) \rightarrow K_+(\mathbf{VF})$*

To obtain the integration result, we need to know the kernel of the map \mathbb{L} from the proposition, i.e., answer the question: which sets in \mathbf{RV} pull back to the same (up to bijection) set in \mathbf{VF} ? This is done in [12, §7] (and is the most technical part of the paper). The answer turns out to be the following: The subset \mathbf{RV}^+ of \mathbf{RV}_0 , consisting of elements x with $v_{\mathbf{RV}}(x) > 0$, pulls back under \mathbb{L} to the open ball of valuation radius 0 around 0 (i.e., to the maximal ideal). On the other hand, the singleton $\{1\} \subseteq \mathbf{RV}$ pulls back to the open ball of the same radius around 1. Hence, their image in $K_+(\mathbf{VF})$ is the same. This turns out to be the only relation: letting I_{sp} be the congruence in $K_+(\mathbf{RV}[*])$ generated by $([\mathbf{RV}^+]_1, [\{1\}]_1)$ (where $[\mathbf{X}]_k$ denotes the class of \mathbf{X} in the k -th filtered piece of the semiring) one obtains the following main result.

Theorem 3.4.6 (Theorem 8.8 of [12]). *The kernel pair of the map $\mathbb{L} : K_+(\mathbf{RV}[*]) \rightarrow K_+(\mathbf{VF})$ is precisely I_{sp} . Hence there is an “integration” map $\int : K_+(\mathbf{VF}) \rightarrow K_+(\mathbf{RV}[*])/I_{\text{sp}}$ of filtered semirings, so that $\int([\mathbb{L}(\mathbf{X})]) = [\mathbf{X}]/I_{\text{sp}}$ for all \mathbf{X} in \mathbf{RV} .*

The next stage is to analyse the target of the integration map, $K_+(\mathbf{RV}[*])/I_{\text{sp}}$. The inclusion of \mathbf{RES} in \mathbf{RV} obviously determines a map from $K_+(\mathbf{RES}[*])$ into $K_+(\mathbf{RV}[*])$. On the other hand, pulling back along $v_{\mathbf{RV}}$ determines a map from definable sets in $\mathbf{\Gamma}$ to definable sets in \mathbf{RV} . To obtain a map on the level of Grothendieck rings, we need to determine which isomorphisms in $\mathbf{\Gamma}$ lift to isomorphisms in \mathbf{RV} .

We have already mentioned that $\mathbf{\Gamma}$ has the structure of a pure divisible ordered abelian group, over the subgroup A of values of the base. Any definable map in this theory is piecewise of the form $\bar{x} \mapsto B\bar{x} + c$, where B is a matrix over \mathbb{Q} , and c is defined over A , i.e., an element of $\mathbb{Q} \otimes A$. However, not every such map lifts to a map in \mathbf{RV} . For instance, the map $x \mapsto \frac{x}{2}$ on $\mathbf{\Gamma}$ corresponds to “extracting a square root” on \mathbf{RV} , so does not exist as a definable map. Likewise, if $c \in \mathbb{Q} \otimes A \setminus A$, then $x \mapsto x + c$ does not lift.

On the other hand, if $B \in \mathbf{GL}_n(\mathbb{Z})$, and $c \in A$, then the map lifts: $\mathbf{GL}_n(\mathbb{Z})$ acts on H^n for any abelian group H , and translation lifts to multiplication by any preimage of c under $v_{\mathbf{RV}}$. It turns out that these are precisely the morphisms that lift; this is already true in the context of §3.3 (and is left as an exercise). This motivates the definition of $\mathbf{\Gamma}[n]$ above. Thus, pullback induces a map of graded semirings $K_+(\mathbf{\Gamma}[*]) \rightarrow K_+(\mathbf{RV}[*])$.

Combining the two maps, we obtain a graded map $K_+(\mathbf{RES}[*]) \otimes_{K_+(\mathbf{\Gamma}[*])} K_+(\mathbf{\Gamma}[*]) \rightarrow K_+(\mathbf{RV}[*])$ (the tensor product is also graded). Let $\mathbf{\Gamma}^{\text{fin}}[n]$ be the full subcategory of $\mathbf{\Gamma}[n]$ consisting of finite subsets. If $\mathbf{X} \subseteq \mathbf{\Gamma}^n$ is a finite subset, then $v_{\mathbf{RV}}^{-1}(\mathbf{X})$ is, by definition, in $\mathbf{RES}[n]$, and its image in $K_+(\mathbf{RV}[n])$ coincides with that of \mathbf{X} . Hence, the map above induces a graded map

$$K_+(\mathbf{RES}[*]) \otimes_{K_+(\mathbf{\Gamma}^{\text{fin}}[*])} K_+(\mathbf{\Gamma}[*]) \rightarrow K_+(\mathbf{RV}[*]) \quad (30) \quad \{\text{E:step2}\}$$

It follows directly from 3.3.2 that this map is surjective. In fact, we have

Theorem 3.4.7. *The map (30) is an isomorphism of graded semirings.*

The injectivity is proved in [12, Prop. 10.2] (and is not difficult). Composing the inverse of this map with the integration map from 3.4.6, we obtain the motivic integration map

$$\int : K_+(\mathbf{VF}) \rightarrow K_+(\mathbf{RES}[*]) \otimes_{K_+(\mathbf{\Gamma}^{\text{fin}}[*])} K_+(\mathbf{\Gamma}[*]) / I_{\text{sp}} \quad (31)$$

In the following section, we will study in more detail the co-domain of this map, and some specialisations.

4. THE CO-DOMAIN OF THE INTEGRATION MAP

In this section we analyse in somewhat more details the graded semiring $K_+(\mathbf{RES}[*]) \otimes_{K_+(\mathbf{\Gamma}^{\text{fin}}[*])} K_+(\mathbf{\Gamma}[*])$ in which the motivic integrals take values. We present certain specialisations in terms of more familiar objects. We conclude with an application from [13] to the monodromy action on the Milnor fibre. We mostly follow §9,10 of [12], as well as [13].

The notation in this section is as follows: K is the ground field, which we assume to be definably closed (i.e., Henselian), the residue field is denoted by \mathbb{k} , and the group of values by A . We recall that the definable closure of A is the divisible hull $\mathbb{Q} \otimes A$ of A .

4.1. Specialising the integration map. We recall that by quantifier elimination, any definable function from $\mathbf{\Gamma}^n$ to $\mathbf{\Gamma}^n$ is piecewise of the form $x \mapsto Mx + a$, where M is a matrix over \mathbb{Q} , and a is a tuple in the definable closure of A . However, not every such map is admitted in the definition of the semi-ring $K_+(\mathbf{\Gamma}_A[*])$ that occurs in the statement of motivic integration: We allow only maps that can be lifted to \mathbf{RV} , namely, those for which M is over \mathbb{Z} , and $a \in A$ (rather than in $\mathbb{Q} \otimes A$).

Thus, we have surjective maps of semi-rings $K_+(\mathbf{\Gamma}_A[*]) \rightarrow K_+(\mathbf{\Gamma}_{\mathbb{Q} \otimes A}[*]) \rightarrow K_+(\mathcal{DOAG}_A)$, where $K_+(\mathcal{DOAG}_A)$ is the usual Grothendieck semi-ring of the theory \mathcal{DOAG}_A of divisible ordered abelian groups over A . We also have the corresponding map of rings, $c : K(\mathbf{\Gamma}_A[*]) \rightarrow K(\mathcal{DOAG}_A)$. We fix A for the moment, and omit it from the notation. Note that under this map, the graded sub-ring $F := K(\mathbf{\Gamma}^{\text{fin}}[*])$ corresponding to finite sets is mapped onto \mathbb{Z} , induced by counting the points.

The ring $K(\mathcal{DOAG})$ is not graded, but the grading can be restored by tensoring with $\mathbb{Z}[\mathbb{T}]$, to obtain a graded map $c_1 : K(\Gamma[*]) \rightarrow K(\mathcal{DOAG})[\mathbb{T}]$, $c_1(x) = c(x)\mathbb{T}^n$ for $x \in K(\Gamma[n])$. We therefore obtain an induced map

$$\tilde{\mathcal{E}} : K(\mathbf{RES}[*]) \otimes_{\mathbb{F}} K(\Gamma[*]) \rightarrow K(\mathbf{RES}[*]) \otimes_{\mathbb{F}} K(\mathcal{DOAG})[\mathbb{T}] \quad (32)$$

(all our maps and tensor products are graded). We note that the image of \mathbb{F} in $K(\mathcal{DOAG})$ is $\mathbb{Z}[\mathbb{T}]$, thus

$$K(\mathbf{RES}[*]) \otimes_{\mathbb{F}} K(\mathcal{DOAG})[\mathbb{T}] = !K(\mathbf{RES}[*]) \otimes_{\mathbb{Z}} K(\mathcal{DOAG}) \quad (33)$$

(with $K(\mathcal{DOAG})$ concentrated in degree 0), where $!K(\mathbf{RES}[*]) = K(\mathbf{RES}[*]) \otimes_{\mathbb{F}} \mathbb{Z}[\mathbb{T}]$ is the quotient of $K(\mathbf{RES}[*])$ by the ideal generated by the relations $[\mathbf{RV}_a] - [\mathbf{RV}_0]$ for $a \in \mathbb{Q} \otimes A$ (and \mathbb{T} is identified with the class of $[\mathbf{RV}_0]$).

According to Theorem 3.4.7, the domain of $\tilde{\mathcal{E}}$ is isomorphic to $K(\mathbf{RV}[*])$. To obtain a map on $K(\mathbf{VF})$, we will compose with a map that kills I_{sp} , the ideal generated by $[1]_1 - c(s)\mathbb{T} - 1$, where $[1]_1$ is the class of $\{1\} \subseteq \mathbb{A}^1$, and s is the class of $(0, \infty) \subseteq \Gamma$ (note that I_{sp} is not homogeneous). Let $u = [1]_1 - c(s)\mathbb{T}$. Then the map that sends $x \in K(\mathbf{RV}[n])$ to $\frac{\tilde{\mathcal{E}}(x)}{u^n} \in (!K(\mathbf{RES}[*]) \otimes K(\mathcal{DOAG}))[\mathbb{u}^{-1}]$ annihilates I_{sp} . Thus we have the following result.

Corollary 4.1.1. *There exists a map*

$$\mathcal{E} : K(\mathbf{VF}) \rightarrow (!K(\mathbf{RES}[*]) \otimes K(\mathcal{DOAG}))[\mathbb{u}^{-1}] \quad (34)$$

where $u = [1]_1 - c([(0, \infty)])\mathbb{T}$, which satisfies $\mathcal{E}(rv^{-1}(\mathbf{X})) = \frac{[\mathbf{X}]}{u^k}$ for $\mathbf{X} \subseteq \mathbf{RES}^k$.

4.2. Computing $K(\mathcal{DOAG})$. It turns out that $K(\mathcal{DOAG})$ can be computed explicitly. Before doing so, we make a general remark: Assume we are given a definable map $f : \mathbf{Z} \rightarrow \mathbf{X}$ and a model M (all for an arbitrary theory \mathcal{T}). Every $a \in \mathbf{X}(M)$ determines an element $[f^{-1}(a)]$ of $K(\mathcal{T}_M)$. If this is the same class c for all a , and c comes from $K(\mathcal{T})$, then $[\mathbf{Z}] = [\mathbf{X}] \cdot c$. In this case, we call f a *definable fibration*. This is also true when working with more restricted classes of “admissible” maps or objects, assuming that an M -definable map is admissible if and only if it is a point in an admissible family (over 0).

Exercise 4.2.1. Prove the remark above

Proposition 4.2.2. $K(\mathcal{DOAG}_A) = \mathbb{Z}^2$, with the class t of $[0, \infty)$ a non-trivial idempotent

Proof. We first note that finite sets are isomorphic in \mathcal{DOAG}_A if and only if they have the same cardinality (since every element is definable). Furthermore, the classes of any two sets of the form $[a, \infty)$ are equal by translating, hence the class of any bounded half-open interval $[a, b)$ is 0, since $[a, \infty) = [a, b) \cup [b, \infty)$. This determines the class of any *bounded* set in one variable. It follows that the class of any 1-variable subset is determined by the value of t , i.e., the subring of $K(\mathcal{DOAG}_A)$ generated by sets in one variable is a homomorphic image of $\mathbb{Z}[t]$.

We now prove by induction on n that the class of a definable set in n variables is generated by classes of 1-variable sets. Consider the projection of a definable subset $Z \subset \Gamma^{n+1}$ to the first n coordinates. By \mathfrak{o} -minimality and compactness, Γ^n can be partitioned into finitely many pieces, such that the fibres over each have the same shape (i.e., a disjoint union of the same number of intervals, each of the same kind). By the one variable case, all such fibres have the same class, hence, by the remark before the proof and the induction hypothesis we are done.

We now show that t is an idempotent: t^2 is the class of $[0, \infty)^2 = \{0\} \times [0, \infty) \dot{\cup} \mathbf{X} \dot{\cup} \mathbf{Y}$, where $\mathbf{X} = \{(x, y) \in (0, \infty)^2 : x \leq y\}$ and $\mathbf{Y} = \{(x, y) \in (0, \infty) \times [0, \infty) : y < x\}$. Again by the remark prior to the claim, the classes of \mathbf{X} and \mathbf{Y} are the classes of bounded half-closed intervals, multiplied by the class of $(0, \infty)$, hence they are 0. Therefore, $t^2 = t$, and $K(\mathcal{DOAG}_A)$ is a homomorphic image of $\mathbb{Z}[t]/t^2 - t = \mathbb{Z}^2$. To show that this is an isomorphism, we need to produce maps $\chi, \chi' : K(\mathcal{DOAG}_A) \rightarrow \mathbb{Z}$ with $\chi(t) = 0$ and $\chi'(t) = 1$.

The map χ can be defined as follows: We may assume that A is countable. Embed A in \mathbb{R} , and set $\chi(\mathbf{X}) = \chi_c(\mathbf{X}(\mathbb{R}))$, where χ_c is the topological Euler characteristic with compact supports. This does not depend on the embedding of A in \mathbb{R} since we have already seen that there is at most one such map. Since topologically, $[0, \infty)$ is homeomorphic to $[0, a)$ for all a , we have $\chi(t) = 0$ (this invariant can also be defined directly in the \mathfrak{o} -minimal setting without passing through the real realisation, cf [9]).

The second invariant is defined by $\chi'(\mathbf{X}) = \lim_{r \rightarrow \infty} \chi(\mathbf{X} \cap [-r, r]^k)$, where k is the ambient dimension. The limit exists by \mathfrak{o} -minimality, and direct calculation shows that $\chi'(t) = 1$. This map is clearly not invariant under all homeomorphisms, but it is invariant under all definable maps in \mathcal{DOAG} , i.e., under \mathbb{Q} -linear maps. To show this, one shows that any compact (in the \mathbb{R} -realisation) definable subset is contractible, hence has Euler characteristic 1, and then induction on dimension (we omit the details, cf. [12, Lemma 9.6]) \square

We denote by \mathbb{L} the class of \mathbb{A}^1 . Recall the difference between $K(\mathbf{RES}[*])$ and $K(\mathbf{RES})$: $K(\mathbf{RES})$ is the (non-graded) quotient of $K(\mathbf{RES}[*])$ in which we forget the ambient dimension. $!K(\mathbf{RES})$ is its quotient identifying all the definable $[\mathbf{RV}_a]$. We may now make Corollary 4.1.1 more explicit (the proof is a direct application of the proposition).

Corollary 4.2.3. *The map \mathcal{E} from Corollary 4.1.1 is identified under 4.2.2 with the pair*

$$(\mathcal{E}_1, \mathcal{E}_2) : K(\mathbf{VF}) \rightarrow !K(\mathbf{RES}[*])[[\mathbb{1}]_1^{-1}] \oplus !K(\mathbf{RES}[*])[\mathbb{L}^{-1}] \quad (35)$$

They induce maps $\overline{\mathcal{E}}_1 : K(\mathbf{VF}) \rightarrow !K(\mathbf{RES})$ and $\overline{\mathcal{E}}_2 : K(\mathbf{VF}) \rightarrow !K(\mathbf{RES})[\mathbb{L}^{-1}]$, and both induces the same map $\mathbf{EU}_\Gamma : K(\mathbf{VF}) \rightarrow \overline{K(\mathbf{RES})}$, where $\overline{K(\mathbf{RES})}$ is the quotient of $!K(\mathbf{RES})$ (and of $!K(\mathbf{RES})[\mathbb{L}^{-1}]$) by \mathbb{T} .

4.3. Computing $K(\Gamma_A^{\text{fin}}[*])$. It is also possible to compute rather explicitly the Grothendieck ring of finite sets in Γ . We begin with a few preliminary remarks. Each of the individual elements of a finite definable set is itself definable (by o-minimality). Therefore, $K_+(\Gamma_A^{\text{fin}}[*])$ is generated (freely) as a semi-group by the singletons. It follows, in particular, that it is a cancellation semi-group, i.e., that the map into the associated group is an embedding. Hence, we may pass to the ring $F = K(\Gamma_A^{\text{fin}}[*])$.

Let $\tau = [0]_1$ be the class of $\{0\} \subseteq \Gamma_A$. It follows from the above that τ is not a zero-divisor in F : if it was a zero divisor, there would be points \bar{x} and \bar{y} , and a \mathbb{Z} -linear map mapping $(0, \bar{x})$ to $(0, \bar{y})$, but then the same map maps \bar{x} to \bar{y} (so they represent the same element in F). In particular, F embeds in the localisation $F[\tau^{-1}]$, a \mathbb{Z} -graded ring. Each graded piece can be mapped by a power of τ to the 0-th one, denoted H_{fin} . Thus, $F[\tau^{-1}] = H_{\text{fin}}[\tau, \tau^{-1}]$, and it remains to describe the ring H_{fin} .

We note that when $A = \mathbb{Q} \otimes A$, i.e., when A is divisible, then all singletons are isomorphic, and we have $H_{\text{fin}} = \mathbb{Z}$. In general, H_{fin} (and F) may be viewed as measuring the “distance” between A and $\mathbb{Q} \otimes A$. We should thus consider the set Ξ_A of groups between A and $\mathbb{Q} \otimes A$, or, equivalently, subgroups of $\mathbb{Q} \otimes A/A$.

Given an element h of H_{fin} , we may evaluate the number of points of h in any such subgroup U . Thus, we have a map from H_{fin} to the ring \mathbb{Z}^{Ξ_A} of \mathbb{Z} -valued functions on Ξ_A .

The set Ξ_A admits a natural topology, as a subspace of the power set of $\mathbb{Q} \otimes A/A$, with the function space topology.

Exercise 4.3.1. Show that the function from Ξ_A to \mathbb{Z} determined by an element of H_{fin} is continuous (where \mathbb{Z} has the discrete topology).

Thus we have a map from H_{fin} to $\mathcal{C}(\Xi_A, \mathbb{Z})$, the ring of continuous functions on Ξ_A .

Proposition 4.3.2. *The map $H_{\text{fin}} \rightarrow \mathcal{C}(\Xi_A, \mathbb{Z})$ is an isomorphism*

Proof. The proof proceeds by identifying Ξ_A with two other spaces. Let X_{fin} be the set of ring homomorphisms $H_{\text{fin}} \rightarrow \mathbb{Z}$. Any element h of H_{fin} can be viewed as a function from X_{fin} to \mathbb{Z} , sending $\phi \in X_{\text{fin}}$ to $\phi(h)$. We topologise X_{fin} with the weak topology making these functions continuous. Given $\phi : H_{\text{fin}} \rightarrow \mathbb{Z}$, let $U_\phi = \{a \in \mathbb{Q} \otimes A : \phi(\frac{[a]}{\tau}) = 1\}$.

Claim. U_ϕ is a subgroup containing A . The map $\phi \mapsto U_\phi$ is a homeomorphism from X_{fin} to Ξ_A .

Proof. If $a \in U_\phi$, any integer multiple of a has the same class as a , so it is also in U_ϕ . If also $b \in U_\phi$, the $\phi(\frac{[a]}{\tau} \frac{[b]}{\tau}) = 1$, but $[a][b] = [a][a+b]$, since the map $(x, y) \mapsto (x, x+y)$ is \mathbb{Z} -linear. Hence also $\frac{[a+b]}{\tau} = 1$, so $a+b \in U_\phi$. The group contains A since the class of all elements in A is τ .

To show the map is a bijection, let U be a subgroup containing A , and define $\phi_U(\frac{X}{\tau^n}) = \#X(U)$. It is clear that this is the inverse of $\phi \mapsto U_\phi$.

To show that $\phi \mapsto U_\phi$ is continuous, it suffices to show that for each $a \in \mathbb{Q} \otimes A/A$, the map $\phi \mapsto U_\phi(a)$ (where U_ϕ is viewed as a characteristic function) is continuous. But this is the map $\phi \mapsto \phi(\frac{[a]}{\tau})$, which is continuous by the definition of the topology. Since both spaces are compact and Hausdorff, this proves the map is a homeomorphism. \square

For any ring R , let $I(R)$ be the boolean algebra of idempotents in R (with meet $(x, y) \mapsto xy$ and join $(x, y) \mapsto x + y - xy$). If $f : R \rightarrow S$ is a map to another ring, the restriction $I(f) : I(R) \rightarrow I(S)$ is a map of boolean algebras. In particular, if S is an integral domain, then $I(S) = 2$, and $I(f)$ is an element of the Stone space of $I(R)$.

In H_{fin} each element $\frac{[a]}{\tau}$ is an idempotent: $[a][a] = [a]\tau$ via the \mathbb{Z} -linear map $(x, y) \mapsto (x, x - y)$. Let B be the boolean subalgebra of $I(H_{\text{fin}})$ generated by such elements. We thus obtain a restriction map $I : X_{\text{fin}} \rightarrow \text{St}(B)$ to the Stone space of B .

Claim. *The map $I : X_{\text{fin}} \rightarrow \text{St}(B)$ is a homeomorphism*

Proof. Injectivity is clear since H_{fin} is generated by B . Surjectivity is seen by extending a map $\psi : B \rightarrow 2$ to H_{fin} via the relation $\psi(a + b) = \psi(a \vee b) + \psi(a \wedge b)$. Continuity follows directly from the definition of the topology. \square

It follows that we need to identify H_{fin} with the continuous functions from $\text{St}(B)$ to \mathbb{Z} . Since $\text{St}(B)$ is compact, each such function has a finite image. Thus, we need to show that $H_{\text{fin}} = \bigcup_n \mathcal{C}(\text{St}(B), [-n, n])$. This follows from Stone duality, since H_{fin} is generated as an abelian group by B . \square

Remark 4.3.3. (1) The proof identifies Ξ_Λ with X_{fin} , the space of \mathbb{Z} -points of the scheme $\text{Spec}(H_{\text{fin}})$, so that H_{fin} is the ring of regular functions.

(2) The map of counting points in a model, used in the previous section, corresponds to the subgroup $\mathbb{Q} \otimes A$. \square

4.4. Computing $K(\mathbf{RES})$. Recall that \mathbf{RES} is a reduct generated by (essentially) 1-dimensional spaces \mathbf{RV}_a , for $a \in \mathbb{Q} \otimes A$, over the residue field \mathbf{RV}_0 . Thus, $K(\mathbf{RES})$ includes $K(\mathbf{RV}_0) = K(\mathcal{V}\text{ar})$, the Grothendieck ring of varieties. However, Unless $a \in A$, the space \mathbf{RV}_a has no point over 0, and so is not isomorphic to the residue field \mathbf{RV}_0 , and so $K(\mathbf{RES})$ is generally bigger. We have also considered the ring $!K(\mathbf{RES})$, in which the classes of these spaces are identified. However, this identification still does not collapse $K(\mathbf{RES})$ into $K(\mathcal{V}\text{ar})$, since there is no reason that other subsets of \mathbf{RES} are identified with subsets of \mathbf{RV}_0^n . Nevertheless, we will identify sets in \mathbf{RES} with varieties equipped with additional data.

Again the difficulties are in the distinction between the (Henselian) ground field K and its algebraic closure. This difference is measured by the Galois group G of K , or, more to the point in this case, by the *inertia subgroup* I : this

inertia subgroup

is the subgroup consisting of elements that act trivially on the residue field \mathbb{k}^a .

We view \mathbf{RV}_a , for $a \in \mathbb{Q} \otimes A$, as a definable 1-dimensional vector space. If a is a tuple $a = (a_1, \dots, a_n)$, we denote by \mathbf{RV}_a the direct sum of the \mathbf{RV}_{a_i} , again a definable vector-space over \mathbf{RV}_0 . We denote by $\mathcal{LR}\mathcal{ES}$ the sub-category of definable sets in \mathbf{RES} consisting of the linear spaces $\mathbf{RV}_{\bar{a}}$ and linear definable maps between them.

The Galois action of I on K^a determines an action of I on each $\mathbf{RV}_a(K^a)$, which, by the definition of I , is linear over $\mathbf{RV}_0(K^a) = \mathbb{k}^a$. It is a basic fact of valuation theory (in equal characteristic 0) that every 1-dimensional representation of I occurs in this way (i.e., $\mathbb{Q} \otimes A/A$ is the dual of I). Since I is pro-finite, every representation is a direct sum of rank-one representations, so the category $\mathcal{R}\text{ep}_I$ of representation of I over \mathbb{k}^a is equivalent to the category $\mathcal{LR}\mathcal{ES}$.

Remark 4.4.1. We have described the equivalence on the level of categories. However, $\mathcal{LR}\mathcal{ES}$ admits a tensor structure, determined by the rule $\mathbf{RV}_a \otimes \mathbf{RV}_b = \mathbf{RV}_{a+b}$ for singletons a, b , and multiplication inside \mathbf{RV} equips the equivalence above with a tensor structure. In the language of Tannakian categories, the assignment $V \mapsto V(K^a)$ determines a fibre functor ω over \mathbb{k}^a , so I can be described, without reference to the Galois group, as the group corresponding to the neutralised Tannakian category $(\mathcal{LR}\mathcal{ES}, \omega)$.

Alternatively, we may describe I in model theoretic language as the binding group corresponding to the internal cover \mathbf{RES} of $\mathbf{RV}_0 = \mathcal{ACF}$. \square

Let $\mathbf{X} \subseteq \mathbf{RV}_{\bar{a}}$ be a definable subset. By quantifier elimination, it corresponds to a constructible subset of the corresponding representation. Conversely, any constructible subset of a representation is definable. In particular, any closed subset can be viewed as an affine variety over \mathbb{k} , together with an action of I . This I -variety satisfies the additional requirement that it embeds (equivariantly) into a representation. However, this is not a restriction:

Lemma 4.4.2. *Let I be an algebraic group acting on a quasi-projective variety \mathbf{X}*

- (1) *If I is finite, then \mathbf{X} can be covered by an invariant affine cover (i.e., by affine I -varieties)*
- (2) *If \mathbf{X} is affine, it can be embedded equivariantly as a closed subvariety of an affine space with a linear action of I .*

sketch of proof. (1) Since I is finite, we need to show: any finite subset of \mathbf{X} is contained in an affine subvariety of \mathbf{X} . This can be done by embedding \mathbf{X} in a projective space, and removing a sufficiently general hyper-plane.

- (2) Let R be the coordinate ring of \mathbf{X} . The action of I on \mathbf{X} induces a linear action on R . Any representation of I is a union of finite dimensional representations, so there is a finite-dimensional subrepresentation V of R that generates R as an algebra. The surjective

map from the symmetric algebra on V to R determines an equivariant closed embedding of \mathbf{X} into the (affine space associated to the) dual representation V^* \square

We denote by $K(\mathcal{V}\text{ar}_{\mathbb{k}}, I)$ the Grothendieck ring of quasi-projective varieties over \mathbb{k} , together with an (algebraic) action of I . As in 4.2.3, we let $\overline{K(\mathbf{RES})}$ be the quotient of $K(\mathbf{RES})$ by all the classes $[\mathbf{RV}_a]$, and similarly, $\overline{K(\mathcal{V}\text{ar}_{\mathbb{k}}, I)}$ is the quotient of $K(\mathcal{V}\text{ar}_{\mathbb{k}}, I)$ by the classes of $\mathbb{A}^1 - \{0\}$, with an arbitrary action. Then the above discussion (together with the lemma) shows the following.

Proposition 4.4.3. *Let I be the inertia group of the base field K . Then $K(\mathbf{RES})$ is isomorphic to the Grothendieck ring $K(\mathcal{V}\text{ar}_{\mathbb{k}}, I)$ of quasi-projective varieties over \mathbb{k} equipped with an action of I . This isomorphism descends to an isomorphism $\Phi_0 : \overline{K(\mathbf{RES})} \rightarrow \overline{K(\mathcal{V}\text{ar}_{\mathbb{k}}, I)}$.*

4.5. The motivic Milnor fibre. As an application, we outline the statement and proof of one of the main results in [13].

Consider again the situation from the beginning of Section 1: We are given a subvariety \mathbf{X}_0 of a smooth variety \mathbf{X} , determined by an equation $f = 0$, where f is a regular function on \mathbf{X} , all over \mathbb{C} . We would like to study the topological properties of a singular point of \mathbf{X}_0 , i.e., a point x where the differential $df(x)$ is zero. We fix all these data; in fact, the whole discussion will be local, so we assume that $\mathbf{X} = \mathbb{A}^n$, so that $f(t_1, \dots, t_n)$ is a polynomial. For example, one could consider $\mathbf{X} = \mathbb{A}^2$, $x = (0, 0)$ and $f(x, y) = y^2 - x^3$, or $\mathbf{X} = \mathbb{A}^3$, $x = 0$, and $f(x, y, z) = x^2 + y^2 - z^2$.

While \mathbf{X}_0 is singular at x , we may hope it looks nicer when slightly deformed. *Milnor's fibration theorem* states that given a sufficiently small B around x , we may find a disc D around 0 in $\mathbb{A}_1(\mathbb{C})$, contained in $f(B)$, such that the restriction of f to $B \cap f^{-1}(D^*)$ is a locally trivial (in fact, smooth) fibration over $D^* = D \setminus \{0\}$. The diffeomorphism type F of each fibre is called the *Milnor fibre* of f at x (it is independent of the choice of B and D).

Milnor fibre

Topologically, D^* is a punctured disc. Pulling back the fibration to the universal cover E of D^* , we obtain a trivial fibration over E , with the same fibre F . Fixing a trivialisation, one obtains an action of the group $G_0 = \text{Aut}(E/D^*)$ on F (changing the trivialisation amounts to conjugation in G_0 , which is trivial since $G_0 = \mathbb{Z}$ is abelian, so the action is well defined). This action is called the *Monodromy action*.

Monodromy action

So we have assigned to the original data a space F , and an action of the group G_0 . The functor of *cohomology with compact supports* assigns, to each space \mathbf{Z} a finite dimensional graded \mathbb{Q} -vector space $H_c^*(\mathbf{Z}, \mathbb{Q})$ (the details of the particular construction of these vector spaces will not be important for us), and the action of G_0 on F determines a representation of G_0 on $H_c^*(F, \mathbb{Q})$. Applying the trace map, we obtain a character $\Lambda : G_0 \rightarrow \mathbb{Q}$. A theorem of Deligne asserts that $\Lambda(n) = 0$ whenever $n > 0$ is less than the *multiplicity* of f at x (this was first proven for $n = 1$ by A'Campo, [2, 1]).

The multiplicity is defined below. At this point we only mention that the action of G_0 on every such space H_c^* is algebraic, and therefore factors through a finite quotient. It follows that we may replace G_0 by its pro-finite completion G . Geometrically, rather than going to the universal cover E of D^* , we pass to the finite (Galois) covers $E_n \rightarrow D^*$, where $E_n = D^*$, and the map is given by $z \mapsto z^n$ (if D is the unit disc). The automorphism group $\text{Aut}(E_n/D^*)$ acts as before, and the advantage is that now every cover is algebraic, and G is the inverse limit of the automorphism groups $\text{Aut}(\mathbb{C}(t^{\frac{1}{n}})/\mathbb{C}(t))$. The character Λ can be viewed as a (continuous) character on G .

4.5.1. *Multiplicity and arc spaces.* It follows directly from the definition of df that for $u + vt \in \mathbb{A}^n(\mathbb{C}[t]/t^2)$, $f(u + vt) = f(u) + df(u)(v)t$. Thus, $u + vt$ is a solution of $f(T) = 0$ in $\mathbb{C}[t]/t^2$ (i.e., a $\mathbb{C}[t]/t^2$ -point of \mathbf{X}_0) if and only if $u \in \mathbf{X}_0(\mathbb{C})$, and v is a tangent vector to \mathbf{X}_0 at u . In particular, u is a singular point of \mathbf{X}_0 precisely when the image under f of the *first infinitesimal neighbourhood* $B_1(x) = \{y \in \mathbf{X}(\mathbb{C}[t]/t^2) : y(0) = x\}$ of x in \mathbf{X} consists of x alone (here $y \mapsto y(0)$ is the map induced by sending t to 0; we will also denote it by $y \mapsto \bar{y}$).

multiplicity

More generally, we may define the n -th infinitesimal neighbourhood of x by $B_n(x) = \{y \in \mathbf{X}(\mathbb{C}[t]/t^{n+1}) : y(0) = x\}$, and the *multiplicity* of f at x as the smallest n for which $f(B_n) \neq 0$. Thus, the multiplicity is 1 if and only if x is a regular point of \mathbf{X}_0 . In the case that f is a polynomial of one variable, it is easily checked that this definition coincides with the usual multiplicity of the zero x of f . We note also that if we define

$$\mathcal{X}_{x,n} = \{y \in \mathbf{X}(\mathbb{C}[t]/t^{n+1}) : y(0) = x, f(y) = t^n\} \quad (36)$$

then the multiplicity is the smallest n for which $\mathcal{X}_{x,n}$ is non-empty.

The collection of algebras $A_n = \mathbb{C}[t]/t^{n+1}$ is equipped with maps $A_n \rightarrow A_{n+m}$, $t \mapsto t^m$, and the sets $\mathcal{X}_{x,n}$ are compatible with these maps: $\mathcal{X}_{x,n}$ maps into $\mathcal{X}_{x,n+m}$. We set $A = \varinjlim A_n$ with respect to these maps. The group G acts on A over A_1 : it acts on A_n through the finite quotient $\text{Aut}(E_n/D^*)$, by restricting the action on the disc to the formal neighbourhood. Thus, this system is dual to the (formal analogue of the) system of topological covers of the punctured disc.

In this correspondence, we may view the elements of $\mathcal{X}_{x,n}$ as global sections of the pullback of the Milnor fibration to E_n : Setting

$$\mathcal{X}_{x,\infty} = \{y \in \mathbf{X}(A) : y(0) = x, f(y) = t \in A_1\} \quad (37)$$

(which can be viewed as the pullback of the Milnor fibre to the universal cover), we see that $\mathcal{X}_{x,n} = \mathcal{X}_{x,\infty}^{H_n}$ (fixed points under H_n), where H_n is the kernel of the map from G to $\text{Aut}(E_n/D^*)$. We set $\mathcal{X}_{x,g} = \mathcal{X}_{x,\infty}^g$ for $g \in G$, so that $\mathcal{X}_{x,n} = \mathcal{X}_{x,g^n}$ for any topological generator g of G . Hence, G/g acts continuously on $\mathcal{X}_{x,g}$.

The set of all A_n -points of \mathbf{X} can be canonically identified with the set of complex points of an algebraic variety $\mathcal{L}_n(\mathbf{X})$, the n -th *truncated arc space*. $\mathcal{X}_{x,n}$ is thus a locally closed subset of $\mathcal{L}_n(\mathbf{X})$, and can be viewed as a topological subspace (with the analytic topology), and we may compute its compactly supported Euler characteristic $\chi_c(\mathcal{X}_{x,n})$, the trace of the identity on H_c^* . The description above, combined with the Lefschetz fixed point principle, provides intuition for the following theorem of Denef and Loeser ([7]).

truncated arc space

Theorem 4.5.2. *Under the above conditions and terminology, for any $g \in G$,*

$$\Lambda(g) = \chi_c(\mathcal{X}_{x,g}) \quad (38) \quad \{\text{E: milnor}\}$$

As mentioned above, for n smaller than the multiplicity, $\mathcal{X}_{x,n}$ is empty, so this theorem implies Deligne's result. Its original proof used explicit calculations, but later Hrushovski and Loeser ([13]) found a proof using motivic integration, which is outlined below.

4.5.3. *Berkovich spaces.* The first step is to replace the complex-analytic spaces that occur in the statement by objects more accessible to the model theoretic methods developed above. This is achieved by replacing the complex-analytic Milnor fibre with the *motivic Milnor fibre* of Nicaise and Sebag ([20]). This is a non-archimedean (Berkovich) analytic space over $\mathbb{C}((t))$. The main relevant facts about these spaces are as follows:

motivic Milnor fibre

- (1) To any definable set \mathbf{U} in \mathbf{VF} over $\mathbb{C}((t))$, one may assign a Berkovich space \mathbf{U}^{an} , and its base-change $\overline{\mathbf{U}^{\text{an}}}$ over $\widehat{\mathbb{C}((t))}^{\text{a}}$, the completed algebraic closure. In particular, this base change admits an action of the Galois group G of $\mathbb{C}((t))$.
- (2) There exists a (compactly supported) cohomology theory that assigns to each Berkovich space X over $\widehat{\mathbb{C}((t))}^{\text{a}}$ a finite-dimensional graded vector space $H_c^*(X, \mathbb{Q}_l)$ over \mathbb{Q}_l (l -adic numbers). This cohomology theory satisfies the standard properties.
- (3) The motivic Milnor fibre mentioned above is determined by the definable set $\mathbf{X}_x = \{y \in \mathbf{X}(\mathcal{O}) : f(y) = t, \bar{y} = x\}$. Its cohomology coincides (as a G -representation) with the cohomology of the topological Milnor fibre (with coefficients in \mathbb{Q}_l).

The combination of the first two properties above determines an additive (and multiplicative) invariant on definable sets in \mathbf{VF} , with values in $K(\text{Rep}_G)$, the Grothendieck ring of the category of l -adic, finite-dimensional graded representations. There is, therefore, an induced map $\text{EU}_{\text{et}} : K(\mathbf{VF}) \rightarrow K(\text{Rep}_G)$. According to the last point, we are interested in the value of EU_{et} on the class of the definable set \mathbf{X}_x .

4.5.4. *Outline of the proof.* The structure of the proof can be described via the following commutative diagram, where $g \in G = \text{Gal}(\mathbb{C}((t)))$, G/g is the quotient of G by the closed subgroup generated by g , and as in Prop. 4.4.3, $\overline{K(\text{RES})}$ is the quotient of $K(\mathbf{RES})$ by all the classes $[\mathbf{RV}_\alpha]$ (for $\alpha \in \mathbb{Q}$, the 0-definable points of Γ), and likewise $\overline{K(\mathcal{V}\text{ar}_{\mathbb{C}}, G)}$ is the quotient of $K(\mathcal{V}\text{ar}_{\mathbb{C}}, G)$

by the classes of $\mathbf{G}_m = \mathbb{A}^1 \setminus \{0\}$, with any action of G (we will use the notation \mathbf{G}_m , despite the fact that the group structure is irrelevant). \mathbf{RES}^g consists of the points of \mathbf{RES} fixed by g (this is a union of definable sets).

$$\begin{array}{ccccc}
\mathbf{K}(\mathbf{VF}) & \xrightarrow{\text{EU}_\Gamma} & \overline{\mathbf{K}(\mathbf{RES})} & \xrightarrow{\cap} & \overline{\mathbf{K}(\mathbf{RES}^g)} \\
& \searrow^{(*)} & \downarrow \wr \Phi_0 & & \downarrow \wr \Phi_g \\
& & \mathbf{K}(\mathcal{V}\text{ar}_{\mathbb{C}}, G) & \xrightarrow{\mathbf{Y} \mapsto \mathbf{Y}^g} & \mathbf{K}(\mathcal{V}\text{ar}_{\mathbb{C}}, G/g) \\
& \searrow^{\text{EU}_{\text{et}}} & \downarrow H_c^* & & \downarrow H_c^* \\
& & \mathbf{K}(\text{Rep}_G) & \xrightarrow{\text{Tr}(g)} \mathbb{Q}_l & \xleftarrow{\chi_c = \text{Tr}(1)} \mathbf{K}(\text{Rep}_{G/g})
\end{array}$$

The isomorphism Φ_0 is the one in 4.4.3 (note that in this case, the inertia group is the whole Galois group G , since the residue field is algebraically closed), and Φ_g is its restriction to $\overline{\mathbf{K}(\mathbf{RES}^g)}$. $\mathbf{Y} \mapsto \mathbf{Y}^g$ is the map induced by associating to a G -variety \mathbf{Y} the sub-variety of g -fixed points (on which G/g then acts continuously). The commutativity of the upper-right square follows directly from the proof of 4.4.3.

H_c^* is the map induced by (étale) cohomology with compact support. This makes sense, since $H_c^*(\mathbf{G}_m) = 0$. The commutativity of the bottom square is a variant of the Lefschetz fixed-point theorem, cf [13, § 5.5] and the reference there.

The commutativity of the triangle $(*)$ is proved in [13, Thm 5.4.1]. The main idea is the following: since both sides are ring homomorphisms, by the main integration results 3.4.6 and 3.4.7, it suffices to prove the statement for (preimages of) elements of $\mathbf{K}(\Gamma)$ and $\mathbf{K}(\mathbf{RES})$ separately. For Γ , the explicit formula for EU_Γ shows that $\text{EU}_\Gamma(b) = 0 \in \overline{\mathbf{K}(\mathbf{RES})}$ for $b \in \mathbf{K}(\Gamma)$ of positive dimension. On the cohomological side, one uses \mathfrak{o} -minimality of Γ to reduce the computation of EU_{et} to computations of cohomology of 1-dimensional annuli, where the result is classical ([13, Lemma 5.4.2]). On the \mathbf{RES} part, one uses again the formula for EU_Γ to note that the class corresponding to a variety over the residue field is simply its pullback to the valued field. The result then follows from specialisation results of Berkovich, along with tracking the action of G (cf. [13, Lemma 5.4.3]).

Given the commutativity of the diagram, the proof proceeds as follows. The motivic Milnor fibre \mathbf{X}_x determines an element of $\mathbf{K}(\mathbf{VF})$. According to the comparison with singular cohomology, the character Λ we are interested in is obtained as $\text{EU}_{\text{et}}([\mathbf{X}_x])$. On the other hand, the space $\mathcal{X}_{x,g}$ is an algebraic variety over \mathbb{C} with an action of G/g , so determines a class in $\overline{\mathbf{K}(\mathcal{V}\text{ar}_{\mathbb{C}}, G/g)} = \overline{\mathbf{K}(\mathbf{RES}^g)}$. Hence, to prove the equation (38), it suffices to prove that \mathbf{X}_x and $\mathcal{X}_{x,g}$ determine the same class in $\overline{\mathbf{K}(\mathbf{RES}^g)}$.

This is done as follows. Consider the set $\mathcal{X}_x = \{y \in \mathbf{X}(\mathcal{O}) : \text{rv}(f(y)) = \text{rv}(t), \bar{y} = x\}$. Let $A' = \mathcal{O}(\mathbb{C}((t))^\alpha)/I$, where I is the ideal of elements with valuation greater than 1. Then $\mathcal{X}_x(\mathbb{C}((t))^\alpha)$ is the pullback, from A' , of the

set of elements $y \in A'$ with $f(y) = t$ and $\bar{y} = x$. But A' is isomorphic to the ring $A = \varinjlim A_n$ considered in 4.5.1, where A_n is identified with $A'_n = \mathbb{C}[t^{\frac{1}{n}}]/t^{1+\frac{1}{n}}$, and the maps in the sequence become inclusions. It follows (using the explicit formula for EU_Γ) that $\mathcal{X}_{x,g} = \mathrm{EU}_\Gamma(\mathcal{X}_x)^g$.

Hence, it suffices to show that $\mathrm{EU}_\Gamma(\mathcal{X}_x) = \mathrm{EU}_\Gamma(\mathbf{X}_x)$. The map $f : \mathcal{X}_x \rightarrow B$, where B is the open ball of radius 1 around t , is a definable fibration, with all fibres isomorphic to the motivic Milnor fibre \mathbf{X}_x . Hence, $\mathrm{EU}_\Gamma(\mathcal{X}_x) = \mathrm{EU}_\Gamma(\mathbf{X}_x) \cdot \mathrm{EU}_\Gamma(B)$. But B is an open ball, so $\mathrm{EU}_\Gamma(B) = 1$, and we obtain the result.

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