SET THEORY

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Abstract. Notes on set theory, mainly forcing. The first four sections closely follow the lecture notes Williams [8] and the book Kunen [4]. The last section covers topics from various sources, as indicated there. Hopefully, all errors are mine.

1. The Suslin problem

1.1. **The Suslin hypothesis.** Recall that $\mathbb{R}$ is the unique dense, complete and separable order without endpoints (up to isomorphism). It follows from the separability that any collection of pairwise disjoint open intervals is countable.

Definition 1.1.1. A linear order satisfies the *countable chain condition* (*ccc*) if any collection of pairwise disjoint open intervals is countable.

Hypothesis 1.1.2 (The Suslin hypothesis)**.** $(\mathbb{R}, <)$ is the unique complete dense linear order without endpoints that satisfies the countable chain condition.

We will not prove the Suslin hypothesis (this is a theorem). Instead, we will reformulate it in various ways. First, we have the following apparent generalisation of the Suslin hypothesis.

Theorem 1.1.3. *The following are equivalent:*

(1) *The Suslin hypothesis*
(2) *Any ccc linear order is separable*

Proof. Let $(\mathbf{X}, <)$ be a ccc linear order that is not separable. Assume first that it is dense. Then we may assume it has no end points (by dropping them). The completion again has the ccc, so by the Suslin hypothesis is isomorphic to $\mathbb{R}$. Hence we may assume that $\mathbf{X} \subseteq \mathbb{R}$, but then $\mathbf{X}$ is separable.

It remains to produce a dense counterexample from a general one. Define $x \sim y$ if both (x, y) and (y, x) are separable. This is clearly a convex equivalence relation, so the quotient $\mathbf{Y}$ is again a linear order satisfying the ccc.

We claim that each class is separable. Indeed, let I_α be a maximal pairwise disjoint family of intervals in the class. Then it is countable, and each I_α is separable.

If $x < y$ in $\mathbf{Y}$, then for any pre-images $\tilde{x}$ and $\tilde{y}$ in $\mathbf{X}$, $(\tilde{x}, \tilde{y})$ is not separable. Let I_k be a maximal collection of pairwise disjoint open sub-intervals (it is countable by the ccc). If the image of each I_k is either x or y, then each is separable, with some dense countable subset D_k. Let $D = \cup_k D_k$. Then D is a countable set, and if it avoids some open interval, then this interval can be added to the I_k, contradicting maximality. □

Definition 1.1.4. A *Suslin order* is a non-separable linear order that has the ccc.

Remark 1.1.5. The proof of Theorem 1.1.3 shows that if a Suslin order exists, then there is also one that is dense, without endpoints, and such no open interval (or convex subset with more than one point) is separable. Hence any such subset is again a Suslin order.

End of lecture 1, Aug 24

1.2. **Ultrametric spaces.**

Definition 1.2.1. Let κ be an ordinal. A (κ-) *ultrametric* on a set $\mathbf{X}$ is a map $v : \mathbf{X}^2 \to \kappa$ satisfying:

(1) $v(a, b) = v(b, a)$
(2) $v(a, c) \geq \min\{v(a, b), v(b, c)\}$
(3) $v(a, a) > v(a, b)$ for any $b \neq a$

A (κ-) *ultrametric space* is a pair $(\mathbf{X}, v)$ with v an ultrametric.

The *resolution* of $\mathbf{X}$ is $\sup\{v(a, a) + 1 | a \in \mathbf{X}\}$ (i.e., it is the smallest ordinal κ for which v is a κ ultrametric)

The weak condition on $v(a, a)$ seems strange, but we will see we need the extra flexibility.

Example 1.2.2. *A usual (discrete) ultrametric space is an $\omega + 1$-ultrametric space with $v(a, a) = \omega$ for all a.*

The geometry makes more sense if we consider the ordinals with the opposite ordering, but we will only do this in our heads.

Exercise 1.2.3. *Show that the usual facts about ultrametric spaces are true. For example:*

(1) *For any $\alpha < \kappa$, the relation $v(a, b) > \alpha$ is a partial (not necessarily reflexive) equivalence relation, and these relations refine each other as α grows. The classes of these relations are called (open)* balls. *They form a basis of open sets for a topology on $\mathbf{X}$. Show that each ball is clopen. We denote by $B(x, \alpha)$ the class of x (i.e., the ball of radius α around x).*
(2) *If $v(a, b) < v(b, c)$ then $v(a, c) = v(a, b)$ (i.e., each triangle is isosceles, and the two equal sides are at least as long as the third one).*
(3) *Conversely, if $\mathbf{X}$ is a set, and E_α for $\alpha < \kappa$ is a sequence of partial equivalence relations such that*
    (a) *E_α refines E_β for $\alpha > \beta$*
    (b) *For any $x \neq y \in \mathbf{X}$, there is $\alpha < \kappa$ such that x and y are not in E_α, but $xE_\alpha x$.*

    Then there is a κ-ultrametric v on $\mathbf{X}$ with E_α induced by v.

Example 1.2.4. *The topology generated by open balls is coarser than the one generated by closed ones. For example, if $\mathbf{X} = \omega + 1$, with $v(a, b) = min(a, b)$ if $a \neq b$, and $v(a, a) = \omega$, then the open balls have the form $B_n = \{i | i > n\}$. In particular, $\omega \in \mathbf{X}$ is not isolated, whereas it is with the closed balls topology.*

Proposition 1.2.5. *Let $(\mathbf{X}, v)$ be an ultrametric space. There is a linear order on $\mathbf{X}$ such that*

(1) *$(\mathbf{X}, v)$ refines the order topology*
(2) *Any ultrametric ball is (order) convex*

Proof. Let $\prec$ be any linear order on the collection of balls. For $x \neq y \in \mathbf{X}$, let $\alpha = v(x,y)$, and define $x < y$ if $B(x,\alpha) \prec B(y,\alpha)$ (note that $x \in B(x,\alpha)$, but the balls are not equal). It is easy to see that $<$ is anti-symmetric and total (in the sense that exactly one of $x < y$ or $y < x$ holds for any $x \neq y$). We note the following property:

If x, y, z are all distinct, and $\alpha = v(z,x) < v(x,y)$ (so $\alpha = v(z,y)$ as well), then either $x < z$ and $y < z$ or $z < x$ and $z < y$. Otherwise, we have (say) $B(x,\alpha) \prec B(z,\alpha) \prec B(y,\alpha)$, but $B(x,\alpha) = B(y,\alpha)$.

It follows from this observation that $<$ is transitive, and so is a total order.

(1) If $x < y$, for any $z \in B(x,\alpha)$ we have $v(z,x) > \alpha = v(x,y)$, so by the observation above, $z < y$. Likewise, any element of $B(y,\alpha)$ is bigger than x. It follows that if $z \in (x,y)$, there is a ball around z contained in (x,y).
(2) Assume $x < z < y$, where x, y both belong to a ball B. It follows from the observation that $v(z,x) = v(y,x)$ or $v(z,y) = v(y,x)$. In either case, $z \in B$. □

End of lecture 2, Aug 26

Exercise 1.2.6. *Show that any linear order as in the Proposition arises in the way it is constructed in the proof (i.e., comes from a linear order on the balls).*

1.2.7. Recall that the *density character* of a topological space is the minimal cardinality of a dense subset (so a space is separable precisely if this is countable). We would like to conclude something about the density character of an ultrametric $\mathbf{X}$.

Lemma 1.2.8. *Let $\mathbf{X}$ be an ultrametric space of resolution κ^+, a successor cardinal. Then the density character of $\mathbf{X}$ is at least κ^+.*

Proof. Let A be a dense subset, and let $\alpha < \kappa^+$. By definition of resolution, there is $x \in \mathbf{X}$ with $v(x,x) > \alpha$. Hence $B(x,\alpha)$ is non-empty, so it contains an element $a \in A$, so $v(a,a) > \alpha$ as well. We find that $\kappa^+ = \cup_{a \in A} v(a,a)$. Since $|v(a,a)| \leq \kappa$ for all a, we get

$$\kappa^+ \leq |A| \cdot \kappa \tag{1}$$

so $|A| \geq \kappa^+$. □

Definition 1.2.9. A *Suslin ultrametric space* is an ultrametric space $\mathbf{X}$ of resolution ω_1 satisfying the ccc.

Proposition 1.2.10. *Any Suslin space $\mathbf{X}$ has a Suslin subspace which is perfect (Recall that a topological space is* perfect *if it has no isolated points.)*

Proof. Let $\mathbf{X}_i$ be a maximal family of pairwise disjoint open balls. Since this is a countable family, at least one of the balls, $\mathbf{Y}$, has resolution ω_1. Clearly, $\mathbf{Y}$ satisfies the ccc, so it is a Suslin space. If it had an isolated point, we could split it away, to get a bigger family of disjoint balls, contradicting maximality. □

Exercise 1.2.11. *Let $(\mathbf{X}, v)$ be an ultrametric space, $x \in \mathbf{X}$. Show that if $v(x,x)$ is not a limit ordinal, then x is isolated. Hence, if $\mathbf{X}$ is perfect, each $v(x,x)$ is a limit ordinal.*

Corollary 1.2.12. *If there is a Suslin ultrametric space, then there is a Suslin order.*

Proof. Let $(\mathbf{X}, v)$ be a Suslin space. By Proposition 1.2.10, we may take it to be perfect. By Lemma 1.2.8, $\mathbf{X}$ is not separable.

Now let $<$ be an order as in Proposition 1.2.5. Since any open interval is v-open, and $(\mathbf{X}, v)$ has the ccc, so does $(\mathbf{X}, <)$. If $A \subseteq \mathbf{X}$ is a countable subset, then it avoids some non-empty open ball B. Since $\mathbf{X}$ is perfect, B contains at least 3 points, $x < y < z$. Hence (x, z) is an open non-empty interval contained in B, so avoided by A. Thus A is not dense in $(\mathbf{X}, <)$. □

Remark 1.2.13. If $\mathbf{X}$ is a perfect ultrametric space that satisfies the ccc, then the resolution of $\mathbf{X}$ is not $\omega_1 + 1$. Otherwise, there is some $x \in \mathbf{X}$ with $v(x, x) = \omega_1$, hence for each $\alpha < \omega_1$, $B(x, \alpha)$ is non-empty, and since $\mathbf{X}$ is perfect, contains some $y_\alpha \neq x$. Working inductively, we find a properly nested sequence of balls around x, of cardinality ω_1. Since each ball is clopen, this violates the ccc.

1.3. **Trees.**

Definition 1.3.1. A *tree* is a partially ordered set $(T, <)$ such that for any $a \in T$, the cut $\ulcorner a \urcorner = \{b \in T | b < a\}$ is well ordered. The order type of $\ulcorner a \urcorner$ is called the *height* $\mathrm{ht}(a)$ of a.

For an ordinal α, the α-th *level* of T is the set T_α of elements of height α.

The *height* of T is the minimal α for which T_α is empty.

A *branch* of T is a maximal linearly ordered subset.

An *antichain* of T is a subset of pairwise incomparable elements.

An element a of a tree thus determines a unique order preserving function $\mathrm{ht}(a) + 1 \to T$, and distinct elements determine distinct functions. We will thus identify a with the function it determines (so $a(\beta)$ makes sense for $\beta \leq \mathrm{ht}(a)$).

Definition 1.3.2. Let κ be a (regular) cardinal.

(1) A κ-*tree* is a tree of height κ such that $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.
(2) A κ-*Aronszajn tree* is a κ-tree such that every branch has cardinality $< \kappa$
(3) A κ-*Suslin tree* is a tree of height κ such that each branch and each antichain has cardinality $< \kappa$.

If κ is omitted, we assume $\kappa = \omega_1$ (in the last two cases).

End of lecture 3, Aug 29

It is clear that any κ-Suslin tree is a κ-Aronszajn tree.

Exercise 1.3.3. *When κ is regular, the condition on the height for Aronszajn (and therefore Suslin) trees can be replaced by $|T| = \kappa$.*

We note that König's Lemma says that there are no ω-Aronszajn trees.

Example 1.3.4. *We let S be the tree of injective functions $t : \alpha \to \omega$ for $\alpha < \omega_1$ (i.e., well ordered subsets of $\mathbb{N}$), ordered by inclusion. This tree has height ω_1, and no uncountable branches, but is not Aronszajn, since $|S_\alpha| = 2^\omega$ for $\alpha \geq \omega$. We will see below that it has an Aronszajn subtree but no Suslin subtree.*

Proposition 1.3.5. *For each tree T of height κ there is an ultrametric space $\mathbf{X}_T$, such that*

(1) *The resolution of $\mathbf{X}_T$ is $\kappa+1$ if T has a branch of length κ, and κ otherwise.*
(2) *If every antichain of T is countable, then $\mathbf{X}_T$ has the ccc*

In particular, if T is a Suslin tree, then $\mathbf{X}_T$ is a Suslin space.

Proof. As a set, $\mathbf{X}_T$ is the set of branches of T. If $x, y \in \mathbf{X}_T$ are two branches, we let $v(x,y) = \{\alpha | x(\alpha) = y(\alpha)\}$ (in particular, we require that both are well defined). It is easy to see that this is an ultrametric.

We note that $v(x,x) = \text{ht}(x)$. We note also that $B(x,\alpha) = \{y | y(\alpha) = x(\alpha)\}$, and that each $t \in T$ determines the set $B_t = \{x | t \in x\}$ of branches passing through t, which is a ball (namely, $B_t = B(x, \text{ht}(t))$ for any branch x passing through t). Clearly, the assignment $t \mapsto B_t$ is weakly order preserving. Conversely, every ball B has the shape B_t for some t, namely $t = x(\alpha)$ where $B = B(x,\alpha)$.

(1) This is obvious from the remark above
(2) If B^α is a collection of disjoint open balls, they can be written as B_{t_α} for some t_α, and the t_α are clearly incomparable

If T is a Suslin tree, then it follows from $\text{ht}(T) = \omega_1$, no uncountable branches and the first part that the resolution of $\mathbf{X}_T$ is ω_1. The second part then says that $\mathbf{X}_T$ has the ccc. □

Finally, we may complete the cycle.

Theorem 1.3.6. *The following are equivalent*

(1) *There is a Suslin order*
(2) *There is a Suslin tree*
(3) *There is a Suslin ultrametric space*

Proof. Two implications were proven in Proposition 1.3.5 and Corollary 1.2.12. We start with a Suslin order $(\mathbf{X}, <)$, and construct a Suslin tree T. As per Remark 1.1.5, we will assume that $(\mathbf{X}, <)$ is dense without end points, and that every non-empty open interval is again a (dense) Suslin order.

We will construct T_α for $\alpha < \omega_1$, by induction on α, with each element (except the root) an open interval of $\mathbf{X}$, ordered by reverse inclusion. We set $T_0 = \mathbf{X}$. For each α, we denote $T_{<\alpha} = \cup_{\beta<\alpha} T_\beta$.

Assuming T_β was constructed for $\beta < \alpha$, let B be a branch of $T_{<\alpha}$, and let $\mathbf{Y} = \cap B$. Clearly $\mathbf{Y}$ is convex. Let I_α be a maximal family of disjoint non-empty open intervals properly contained in $\mathbf{Y}$. These are the elements of T_α that prolong B.

We set $T = T_{<\omega_1}$. Clearly T is a tree with levels T_α, so $\text{ht}(T) \leq \omega_1$. Any antichain is a collection of pairwise disjoint non-empty open intervals, so is countable. Likewise, for any branch $\{I_\alpha\}$, $I_\alpha \setminus I_{\alpha+1}$ contains an interval (since the order is dense), and all these intervals are pairwise disjoint, so such a branch is countable as well.

To show that there is an element at each countable height, it is enough to show that at each stage α, there is a branch B as above that contains an open interval. Let A be the set of end-points of the elements of $T_{<\alpha}$. This is a countable set, and $\mathbf{X}$ is not separable, so there is an interval I that avoids it. Hence, for any element J of $T_{<\alpha}$, either $I \subset J$ or $I \cap J = 0$. The set $B = \{J \in T_{<\alpha} | I \subset J\}$ is clearly a chain. To show that it is a branch, let $\beta < \alpha$ be the first such that $B \cap T_\beta = 0$. Then B is a branch of $T_{<\beta}$, and $I \subset \cap B$ is an open interval that does not intersect any element of T_β. This contradicts the construction of T_β. □

End of lecture 4, Aug 31

Remark 1.3.7. The passage from Suslin spaces to Suslin trees can also be done directly: Given a space $\mathbf{X}$, let T be the set of balls, ordered by reverse inclusion. The proof that this tree has the required properties is similar to the above.

1.3.8. On the other hand, Aronszajn exist unconditionally. We make the following (provisional) definitions. Let T be a tree of height κ. We call an equivalence relation $\sim$ on T a *tree relation* if $s \sim t$ implies $s' \sim t'$ for $s' \leq s$ and $t' \leq t$. Given such a relation, a *near branch* of T is a sequence $s_\alpha \in T_\alpha$ (one for each $\alpha < \kappa$) of equivalent elements. We say that $\sim$ is *thin* if the intersection of each level with each class has cardinality less than κ.

Given a near branch s_α, the set $T' = \{t \in T | t \sim s_\alpha\}$ is a subtree of height κ. If $\sim$ is thin, T' is thus a κ-tree. If T had no κ-branches, then T' is κ-Aronszajn.

Theorem 1.3.9. *There is an Aronszajn tree.*

Proof. According to the discussion above, it is enough to define a thin tree equivalence relation on the tree S of Example 1.3.4, and find a near branch. We define $s \sim t$ if the differ only on a finite subset of their common domain. Since each domain is countable, this relation is thin.

We construct an element $s_\alpha \in S_\alpha$ by induction on α, such that the s_α form a near branch, and the image of each s_α is co-infinite.

Set $s_0 = 0$, and let $s_{\alpha+1}$ be any extension of s_α to $\alpha+1$. For α a limit ordinal, let α_i be a cofinal sequence in α, indexed by ω. Set $t_0 = s_{\alpha_0}$, and define by induction a chain $t_n : \alpha_n \to \omega$ such that $t_n \sim s_{\alpha_n}$ and t_{n+1} misses the first $n+1$ elements of $\omega \setminus t_n(\alpha_n)$ (note that $t_n \sim s_n$, so the latter set contains infinitely many elements). Then $s_\alpha = \cup_i t_i$. □

Exercise 1.3.10. *Verify that it is possible to construct the t_i as above.*

Remark 1.3.11. The tree constructed above is not a Suslin tree. In fact, no subtree of S is a Suslin tree. Indeed, if T is a subtree of height ω_1, and n is a natural number, the set A_n of elements of T that have a last element n is an anti-chain. Each $T_{\alpha+1}$ intersects some A_n, so at least one of them must be uncountable.

1.4. **The diamond principle.** We now look for more general set theoretic assumptions that will imply (in particular) the existence of a Suslin tree. Our strategy will be to build a tree by levels T_α, trying to ensure that T is a Suslin tree. The first step is to translate the requirements on T to conditions on the stages T_α.

Exercise 1.4.1. *A tree T is said to be* ever-branching *if for every $a \in T$, the set $T_a = \{b \in T | a \leq b\}$ is not a linear order. Show that if T is ever-branching, then $\mathbf{X}_T$ is perfect. Hence, by Remark 1.2.13, an ever-branching tree of height ω_1 that has no uncountable anti-chains is a Suslin tree.*

Hence, if we have a tree T such that T_α is non-empty if and only if α is countable, and for each $a \in T_\alpha$ there is $\beta > \alpha$ and $b, c \in T_\beta$ with $a < b, c$, then T is a Suslin tree provided it has no uncountable anti-chains. We now reformulate this as a condition on the levels. For any subset A of T, we let $A_\alpha = A \cap T_\alpha$ and $A_{<\alpha} = A \cap (\cup_{\beta<\alpha} T_\beta)$.

Lemma 1.4.2. *Let T be an ω_1-tree with a maximal uncountable chain A. Then the set of limit ordinals $\alpha < \omega_1$ for which $A_{<\alpha}$ is a maximal anti-chain in $T_{<\alpha}$ is cofinal in ω_1.*

The proof is part of a general principle, explained later. We note that the Lemma can be interpreted as a largeness statement: If A is a maximal uncountable anti-chain, then A_α is maximal for a large amount of the α-s. This will also be made precise below.

1.4.3. Assume now that we are building T inductively. It is easy to make sure at each stage that the outcome is an ever-branching ω_1-tree. Let $\alpha < \omega_1$ be a limit ordinal, assume we have constructed $T_{<\alpha}$ and that $D \subseteq T_{<\alpha}$ is a maximal anti-chain. When constructing T_α, we make sure that each element we add is below an element of D (exercise: this can be done). Hence D does not extend to an uncountable anti-chain. By the Lemma, if we can do this for all such D, we have constructed a Suslin tree.

The problem with this approach is that the tree is not yet constructed, so at each limit stage α we do not know which anti-chain to eliminate in this manner. A *diamond sequence* will provide this information in advance. In fact, it will code an element in an arbitrary sequence of countable subsets of ω_1, provided that the sequence is large, in the same sense as above. End of lecture 5, Sep 2

1.4.4. *Filters and ideals.* Recall that a *filter* on a set B is a proper non-empty subset of $\mathcal{P}B$ closed upwards and under finite intersections. A filter on B can be thought of as defining a notion of largeness on subsets of B: a subset A is *$\mathcal{F}$-large* if it is in $\mathcal{F}$. Likewise A is *small* if the complement A^c is large (note that small is not the opposite of large — we also have intermediate size). A set of the form $\mathcal{F}^* = \{A^c | A \in \mathcal{F}\}$ when $\mathcal{F}$ is a filter is called an *ideal*. We call a set *$\mathcal{F}$-stationary* if it is not small.

Examples of filters include the collection of measure 1 subsets of a probability space, or the collection of sets with dense interior in a topological space. The stationary sets will then correspond to positive measure sets, or somewhere dense subsets, respectively.

Exercise 1.4.5. *Show that, given a filter $\mathcal{F}$ on a set X, there is an ordered set $(M, <)$ with least element 0 and greatest element 1, and a "measure" map $\mu : \mathcal{P}X \to M$ (with natural properties), such that $\mu(A) = 1$ iff $A \in \mathcal{F}$, and $\mu(A) > 0$ iff A is stationary.*

Definition 1.4.6. Let κ be a regular cardinal.

(1) A subset $C \subseteq \kappa$ is *closed* if for any increasing bounded sequence a_α of elements of C, $\sup_\alpha a_\alpha \in C$. (In other words, it is closed in the order topology)
(2) A cofinal closed subset is called a *club*.
(3) The collection of subsets of κ containing a club is called the *club filter* (we will see immediately that it is a filter).

The notion of a club is easy to work with because of the following fact, which is a variant of the Downward Löwenheim–Skolem theorem.

Theorem 1.4.7. *Let κ be a regular uncountable cardinal, and let M be a structure in a language of cardinality $< \kappa$, whose universe is κ. Then the set C of $\alpha < \kappa$ that are substructures of M is a club. So is the set of elementary sub-models.*

Proof. The second statement follows from the first by adding Skolem functions, so that any substructure is an elementary sub-model.

It is obvious that C is closed. Given $\alpha < \kappa$, the substructure generated by α still has cardinality $< \kappa$, hence (since κ is regular), it is contained in some ordinal $g(\alpha) < \kappa$. Then $\beta = \sup_i g^i(\alpha)$ is an element of C greater than α. □

Exercise 1.4.8. *Let us call a club on κ* elementary *if it arises as in the theorem (for a structure in some language), and a subset of κ elementary if it contains an elementary club. Show that an intersection of elementary clubs $\cap_{i<\alpha}C_i$ with $\alpha < \kappa$ is an elementary set. In particular, the elementary sets form a filter.*

Proof of Lemma 1.4.2. Let T and A be as in the Lemma. We may assume that as a set, $T = \omega_1$. Consider T as a structure in the language with the tree relation and a unary predicate for A. If α is an elementary sub-model of T, $\alpha \cap A$ is a maximal anti-chain (since this is first order).

Let C be the club of elementary sub-models. Since each $T_{<\alpha}$ is countable, there is a $\beta_\alpha \in C$ containing $T_{<\alpha}$. Likewise, since each $\beta \in C$ is countable, there is an α_β such that $\beta \subseteq T_{<\alpha}$. We thus get a chain

$$\beta_1 \subseteq T_{\alpha_1} \subseteq \beta_2 \subseteq \dots \qquad (2)$$

Where $\beta_i \in C$. The union of this chain is of the form $T_{<\alpha}$ for some α, but is also an elementary sub model. □

Remark 1.4.9. More generally, we have proved the following. Assume that $\mathcal{B}$ is a collection of subsets of κ such that

(1) Each member of $\mathcal{B}$ has cardinality less than κ.
(2) $\mathcal{B}$ is closed under bounded unions
(3) $\cup\mathcal{B} = \kappa$

Then the set of $\alpha \in \kappa$ such that α is an elementary sub-model and $\alpha \in \mathcal{B}$ is a club.

End of lecture 6, Sep 5

Exercise 1.4.10. *Let $f : \kappa \to \kappa$ be a continuous function such that $f(x) \geq x$ for all $x \in \kappa$. Show that the set of fixed points $f(x) = x$ is a club. Conversely, show that any club is the set of fixed points of some f as above*

Lemma 1.4.11. *The intersection of less than κ clubs in κ is a club (in other words, the club filter is a κ-*complete *filter).*

Proof. Let f_α be a function corresponding to the club C_α, as in the exercise. Then the function $g = \sup_{\alpha<\beta} f_\alpha$ is also as in the exercise (it is well defined since $\beta < \kappa$), and the set of fixed points of g is $\cap_\alpha C_\alpha$. □

1.4.12. We explain how these notions can be used to produce many non-isomorphic dense linear orders of cardinality ω_1. We will consider DLOs $(D, <)$ that can be presented as a union $D = \cup_{\alpha<\omega_1} D_\alpha$, where D_α is a countable dense linear order, closed downwards in $D_{\alpha+1}$ (not every order can be presented this way). Given such a presentation $\mathcal{D}$, write $A_\mathcal{D}$ for the set of α such that D_α has a sup in $D_{\alpha+1}$. We note that every D_α is a cut in D, so this is equivalent to having a sup in D.

Claim 1.4.13. *Let D and E be DLOs with presentations $\mathcal{D}$ and $\mathcal{E}$. Then D and E are isomorphic if and only if $B = A_\mathcal{D} \triangle A_\mathcal{E}$ is small (in particular, any two presentations of the same order are "the same up to measure 0").*

Proof. Let $f : D \to E$ be an isomorphism. The set of $\alpha < \kappa$ such that $f(D_\alpha) = E_\alpha$ is an (elementary) club. If B has positive measure, some $\alpha \in B$ is in the club. But this is impossible, since precisely one of D_α and E_α is the cut of an element.

Conversely, if B has measure 0, then there is some club C that it avoids. The club tells us how to construct an isomorphism. □

Exercise 1.4.14. *Given $X \subseteq \omega_1$, the game G_X is played by two players α and β, alternating in choosing a sequence $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \omega_1$ (integer indexed). Player α wins in $\sup \alpha_i \in X$. Determine for which sets X each of the players has a winning strategy.*

Definition 1.4.15. A $\diamondsuit$-*sequence* is a sequence $A_\alpha \subseteq \alpha$ for $\alpha < \omega_1$, such that for any $X \subseteq \omega_1$, the set $\{\alpha \in \omega_1 | X \cap \alpha = A_\alpha\}$ is stationary.

The *Diamond Principle* states that there is a diamond sequence.

Thus, a diamond sequence predicts, ones and for all, the restriction of an arbitrary subset of ω_1, at least with "non-zero probability".

Proposition 1.4.16. *The Diamond Principle implies $\mathcal{CH}$*

Proof. The set of infinite countable ordinals is a club, so for any subset X of ω, there is such an ordinal α with $X = X \cap \alpha = A_\alpha$. □

We may now carry out the plan introduced in the beginning of the section.

Theorem 1.4.17. *The Diamond Principle implies the existence of a Suslin tree.*

Proof. Let A_α be a $\diamondsuit$-sequence. We will build by induction $T = \cup_\alpha T_\alpha$ with universe ω_1, such that each element has at least two descendants, and if A_α is a maximal anti-chain of $T_{<\alpha}$, it remains maximal in $T_{<\alpha+1}$ (hence in T). If B is an anti-chain of the resulting tree, the set of α such that $B_{<\alpha}$ is a maximal anti-chain of $T_{<\alpha}$ is a club, so $A_\alpha = B_{<\alpha}$ for some α. Hence B is countable, and T is a Suslin tree.

We set $T_\alpha = \{\alpha \cdot \omega + n | n \in \omega\}$, with $(\alpha+1) \cdot \omega + 2n$ and $(\alpha+1) \cdot \omega + 2n + 1$ below $\alpha\omega + n$. We only need to define the tree structure on limit ordinals. We assume that α is a limit ordinal with A_α a maximal anti-chain (the other case is easy). Let a_i enumerate $T_{<\alpha}$ ($i < \omega$), and let B_i be a branch of $T_{<\alpha}$ through a_i that intersects A_α, of order type α (such a branch exists by induction and maximality of A_α). We set $x < \alpha\omega + n$ if and only if $x \in B_i$. This determines a tree structure making T_α the α-th level. □

End of lecture 7, Sep 7

Remark 1.4.18. Applying the Diamond condition to elementary clubs, we get the following interpretation: Assume we have a $\diamondsuit$-sequence A_α. Let M be a structure with universe ω_1, and let A be a definable subset. Then there is a limit $\alpha \in \omega_1$ which is an elementary sub-model, with $A \cap \alpha = A_\alpha$. The preceding argument is an application of this principle, where A is a maximal antichain.

Proposition 1.4.19. *The Diamond Principle is equivalent to either of the following statements.*

1. *There is a sequence $R_\alpha \subseteq \alpha \times \alpha$ such that for all $R \subseteq \omega_1 \times \omega_1$, $\{\alpha \in \omega_1 | R \cap \alpha \times \alpha = R_\alpha\}$ is stationary.*
2. *The same but R and R_α are functions.*
3. *For any countable A, there is a sequence $R^A_\alpha \subseteq A \times \alpha$ such that for all $R \subseteq A \times \omega_1$, $\{\alpha \in \omega_1 | R \cap A \times \alpha = R^A_\alpha\}$ is stationary.*

Proof. Let $f : \omega_1 \to \omega_1 \times \omega_1$ be any bijection. The set of α such that f restricts to a bijection from α to $\alpha \times \alpha$ is an elementary club. Thus we have (on a measure 1 subset) a translation between the corresponding subsets. The rest is similar. □

Definition 1.4.20. A $\diamondsuit^-$*-sequence* is a sequence $A_\alpha \subseteq \mathcal{P}\alpha$ for $\alpha < \omega_1$, such that each A_α is countable, and for every $X \subseteq \omega_1$, the set $D_X = \{\alpha \in \omega_1 | X \cap \alpha \in A_\alpha\}$ is stationary.

A $\diamondsuit^+$*-sequence* is a $\diamondsuit^-$-sequence, where in addition the set D_X contains a club C, and furthermore, D_C also contains C.

The $\diamondsuit^+$*-principle* says that there is a $\diamondsuit^+$-sequence.

Proposition 1.4.21. *If there is a $\diamondsuit^-$-sequence, then there is a $\diamondsuit$-sequence*

Proof. As in Proposition 1.4.19, we may assume we have a sequence of countable sets $R_\alpha \subseteq \mathcal{P}(\omega \times \alpha)$ such that for any $R \subseteq \omega \times \omega_1$, the set $\{\alpha \in \omega_1 | R \cap \omega \times \alpha \in R_\alpha\}$ is stationary. We denote by R^k_α the k-th relation in R_α (in some ω-ordering of it). We set $A^k_\alpha = R^k_\alpha(k)$. We claim that for some k, A^k_α is a diamond sequence.

Otherwise, for any k there is a subset $X(k) \subseteq \omega_1$ such that $\{\alpha \in \omega_1 | X(k) \cap \alpha = A^k_\alpha\}$ is small. Since a countable union of small sets is still small, we get that the set $\{\alpha \in \omega_1 | \exists k(X(k) \cap \alpha = R^k_\alpha(k))\}$ is small. However, the choice of R_α is such that $\{\alpha \in \omega_1 | \exists k(X \cap \alpha = R^k_\alpha)\}$ is not small. Since this set is contained in the previous one, we get a contradiction. □

End of lecture 8, Sep 9

It turns out that $\diamondsuit^+$ is stronger than $\diamondsuit$ (but still consistent).

1.5. **Special and $\mathbb{R}$-embeddable trees.** Two lectures by Graham Leach–Krouse. See separate notes by Graham.

End of lectures 9–10, Sep 12, 14

2. Martin's Axiom

2.1. **Omitting types.** We start with a classical theorem. Recall that a type in a first order theory is *isolated* if it is not implied by a formula (in other words, it is an isolated point of the type space). A model omits a type if it does not realise it.

Theorem 2.1.1 (Omitting types). *Let $\mathcal{T}$ be a countable theory, and let p be a non-isolated type. Then $\mathcal{T}$ has model that omits p.*

Proof. By Skolemising, we may assume that there are constants c_i that enumerate a model. Hence we need to find a collection of formulas ϕ_i, each contradicting p, such that $\{\phi_i(c_i)\}$ is consistent with $\mathcal{T}$. Let D be the set of formulas inconsistent with p, and let $D_i = \{\phi(c_i) | \phi \in D\}$ (so we need a theory that contains at least one element of each D_i). The condition that p is not isolated means that for each sentence ψ consistent with $\mathcal{T}$ and each i, there is an element of D_i implying ψ. In the language introduced below, each D_i is dense in the partial order on formulas given by implication, and a theory is a filter in the same order. Hence the Theorem will follow from Proposition 2.1.6. □

We wish to abstract the situation in the proof of the Theorem.

Definition 2.1.2. Let $(P, <)$ be a partial order (not required to by anti-symmetric).

(1) A subset A of P is *dense* if for any $x \in P$ there is $a \in A$ such that $a \leq x$.
(2) $x, y \in P$ are *compatible* if there is $z \leq x, y$. An *antichain* is a subset of pairwise incompatible elements
(3) P satisfies the *countable chain condition* if every anti-chain is countable
(4) A *filter* in P is a non-empty upward closed sub-order in which any two elements are compatible

(5) Given a collection D of subsets of P, a filter is called *D-generic* if it intersects every element in D.

Remark 2.1.3. A filter on a set A is the same as a filter in the sense above on the poset $\mathcal{P}A \setminus 0$ ordered by inclusion (more generally, the same holds with a boolean algebra). Two subsets are incompatible if they are disjoint.

Example 2.1.4. Let P be the collection of open sets in a topological space, ordered by inclusion. Two sets are incompatible if they are disjoint, the ccc and filters have the usual meaning.

In particular, if the space is compact, there is a $D = \{D_\alpha\}$-generic filter if and only if $\cap_\alpha X_\alpha$ is non-empty, where $X_\alpha = \cup D_\alpha$ (recall that in a compact space, the map that send a point to its ultra-filter is a bijection).

Remark 2.1.5. Given $(P, <)$ we may declare the downwards closed subsets to be open. For any $p \in P$, the assignment $p \mapsto U_p = \{q \in P | q \leq p\}$ is an order preserving map from P to the poset of open sets (which is injective precisely if $<$ is anti-symmetric). Furthermore, open sets of this form are a basis for the topology. With this topology and identification, the terminology has its usual meaning (for instance, a subset is dense iff it is topologically dense, p and q are incompatible iff U_p and U_q are disjoint, etc.).

Proposition 2.1.6. *Let $(P, <)$ be a partial order, and let $\{D_i\}$ be a countable collection of generic sets. Then there is a $\{D_i\}$-generic filter.*

Proof. We define finite subsets F_i of P by induction. Assuming F_i was defined, let x be an element smaller than all elements of F_i. Since D_{i+1} is dense, it contains an element $y < x$. Let $F_{i+1} = F_i \cup \{y\}$. Let F be the upward closure of $\cup_i F_i$. Clearly F is a generic filter as required. □

We now wish to prove a stronger version of the omitting types theorem: We would like to omit a set X of types, each non-isolated. For each $p \in X$ and each constant symbol c, we construct, as before, a dense set D_c^p of sentences obtained by substituting c in an arbitrary formula that avoids p. If X is countable, we have a countable set of generic sets, so we get the result from Proposition 2.1.6. Otherwise, we need the following condition.

Definition 2.1.7. Let κ be a cardinal. *Martin's axiom for κ* ($\mathcal{MA}(\kappa)$) is the statement that in any order with the ccc there is a generic filter for any collection of size κ of dense sets.

Martin's axiom is $\mathcal{MA}(\kappa)$ for all $\kappa < 2^\omega$.

Thus, Proposition 2.1.6 says that $\mathcal{MA}(\omega)$ holds, and the argument above shows:

Corollary 2.1.8. *Assume $\mathcal{MA}(\kappa)$. Then given a collection of κ non-isolated types in a countable theory, there is a model that omits all of them.*

We note that the ccc condition holds in particular if P is countable, and in the context of the omitting types theorem it is essentially equivalent to the theory being countable. We discuss the weaker condition, in which P itself is required to be countable, in 4.2.

Remark 2.1.9. Instead of Skolemising, we could reduce the proof completely to Martin's axiom. To do that, we just notice that for any sentence ψ, the set $\{\phi | \phi \rightarrow$

ψ or $\phi \to \neg\psi\}$ is dense, and filter that intersects it will have an opinion on ψ. Hence a theory that intersects all such (countably many) sets is complete. Likewise, for any satisfiable formula $\psi(x)$, the set $\{\psi(c)|c \in C\}$, where C is a countable set of new constants is dense, and in a theory that intersects all of these, the constants C enumerate a model.

Corollary 2.1.10. *$\mathcal{MA}(2^\omega)$ is false*

Proof. Otherwise we could omit all non-isolated types of any countable theory, but there are countable theories with no atomic models (an explicit example is below). □

2.2. **More examples.**

2.2.1. We say that a function $g : \omega \to \omega$ *dominates* another such function f if $g(n) > f(n)$ for almost all n. Given a collection $\mathscr{F}$ of such functions, we would like to know when is there a function that dominates all functions in $\mathscr{F}$.

Proposition 2.2.2. *Assume $\mathcal{MA}(\kappa)$. Then for any set $\mathscr{F} \subseteq \omega^\omega$ of cardinality κ, there is a function g that dominates all $f \in \mathscr{F}$.*

End of lecture 11, Sep 16

Proof. Let $P = \{(g_0 : n \to \omega, \mathscr{F}_0)|\mathscr{F}_0 \subseteq \mathscr{F}, |\mathscr{F}_0| < \omega\}$. We order P by $(g_1, \mathscr{F}_1) < (g_2, \mathscr{F}_2)$ if g_1 extends g_2, $\mathscr{F}_1$ contains $\mathscr{F}_2$, and $g_1(n) > f(n)$ for all $f \in \mathscr{F}_2$ and any n where g_1 is defined and g_2 is not. Intuitively, $(g_0, \mathscr{F}_0)$ represents the condition "g is an extension of g_0 that dominates all elements of $\mathscr{F}_0$ outside the domain of g_0", and the conditions are ordered by implication (see below for more details).

$(g_1, \mathscr{F}_1)$ and $(g_2, \mathscr{F}_2)$ are compatible if for any n, either $g_1(n) = g_2(n)$ or, if $g_1(n)$ is defined, then $g_1(n) > f(n)$ for all $f \in \mathscr{F}_2$ (and conversely). In particular, a filter G determines a partial function g that dominates, on its domain, any function that appears in one of the elements of G. Also, since compatibility depends only on a finite part of the domain, P has the ccc.

To make the function g total, we note that the set D_n of elements of P where the function part is defined on n is dense. A filter that intersects each determines a total function. Likewise, to dominate all functions in $\mathscr{F}$, we note that for $f \in \mathscr{F}$, the set of all elements of P where f appears is dense. Now $\mathcal{MA}(\kappa)$ gives us a filter that intersects each. □

Here is a logical reformulation of the last proof. We work in a language containing a function symbol for each $f \in \mathscr{F}$, a constant symbol for each $n \in \omega$, and the relation $<$, all interpreted in the obvious way (call this theory $\mathcal{T}_0$). We now expand the language by an extra function symbol g, and consider the set P of sentences that are (finite) consistent conjunctions of sentences of the form $\forall x > n(g(x) > f(x))$ and $g(m) = n$, ordered by logical implication (wrt $\mathcal{T}_0$). The truth value of each such sentence depends on a finite set of integers, so P has the ccc. For the same reason, each set $D_f = \{\forall x > n(g(x) > f(x))|n \in \omega\}$ and $D_m = \{g(m) = n|n \in \omega\}$ are dense. Hence there is a consistent extension $\mathcal{T}$ of $\mathcal{T}_0$ containing an element of each D_f and each D_m. In a model of $\mathcal{T}$, g will map ω into ω, and will dominate each f.

Here is a relation to Suslin trees.

Proposition 2.2.3. *$\mathcal{MA}(\omega_1)$ implies that there is no Suslin tree.*

Proof. Assume T is well-pruned Suslin tree, considered with the opposite order. Let D_α for $\alpha < \omega_1$ be the set of nodes of level at least α. Since T is a well-groomed ω_1 tree, each D_α is dense. Since T has the ccc, we get from $\mathcal{MA}(\omega_1)$ a filter that intersects each. Such a filter is a branch of height ω_1. □

Next, we have mad families:

Definition 2.2.4. A family $A \subseteq \mathcal{P}(\omega)$ is *almost disjoint* if each element is infinite, but the intersection of each two is finite.

A *mad family* is a maximal almost disjoint family.

Proposition 2.2.5. *There is a mad family of size continuum, but not of size ω.*

Proof. The set of branches of the complete binary tree is an almost disjoint family of size continuum. Assume $\{A_i\}$ is a countable almost disjoint family. Choose a_i not in A_j for $j < i$, and let $A = \{a_i\}$. Then $\{A_i\} \cup \{A\}$ is also almost disjoint. □

End of lecture 12, Sep 19

Proposition 2.2.6. *$\mathcal{MA}(\kappa)$ implies that any mad family has cardinality bigger than κ*

The Proposition follows from the following more general statement, which will have additional use.

Theorem 2.2.7. *Let $\mathscr{A}, \mathscr{C} \subseteq \mathcal{P}\omega$ be sets of cardinality at most κ, and assume that for any $c \in \mathscr{C}$ and any finite $\mathscr{F} \subseteq \mathscr{A}$, $c \setminus \cup\mathscr{F}$ is infinite. Then $\mathcal{MA}(\kappa)$ implies that there is $R \subseteq \omega$ with $R \cap a$ finite but $R \cap c$ infinite for all $a \in \mathscr{A}$ and $c \in \mathscr{C}$.*

Proof. Let T be the diagram of $\mathscr{A}$ in the language with a unary relation symbol R_a for $a \in \mathscr{A}$, and constant symbol for each element of ω, and expand the language by an additional unary symbol R.

Let P_a, for $a \in \mathscr{A}$ be the set of sentences of the form $\forall x > n(\neg R_a(x) \vee \neg R(x))$ (where $n \in \omega$), and let $\mathbb{P}$ be the set of consistent conjunctions of all such sentences, as well as sentences of the form $R(n)$, for $n \in \omega$ (ordered by implication).

$\mathbb{P}$ has the ccc: For any ϕ in $\mathbb{P}$, set $X_\phi = \{n \in \omega | \phi \to R(n)\}$. Then each X_ϕ is finite, and if $X_\phi = X_\psi$, then ϕ and ψ are consistent.

Let D_a, for $a \in A$, be the set of elements of $\mathbb{P}$ that imply some element in P_a, and let ${E_c}^n$ be the set of elements of $\mathbb{P}$ that imply some $R(m)$ for $m \in c \setminus n$. The condition on $\mathscr{C}$ and $\mathscr{A}$ implies that each of them is dense. Hence, if $\mathcal{MA}(\kappa)$ holds, we may find a consistent theory that contains an element of each D_a and each ${E_c}^n$. In any model M, $R(M) \cap \omega$ is an element as required. □

Corollary 2.2.8. *$\mathcal{MA}(\kappa)$ implies that $2^\kappa = 2^\omega$*

Proof. Let $\mathscr{A}$ be an almost disjoint family of size κ. Define $f : \mathcal{P}(\omega) \to \mathcal{P}(\mathscr{A})$ by sending S to the collection $\mathscr{A}_S$ of sets almost disjoint from S. If $\mathscr{B} \subset \mathscr{A}$, then $\mathscr{C} = \mathscr{A} \setminus \mathscr{B}$ and $\mathscr{B}$ satisfy the conditions of the Theorem. Hence f is onto. □

Corollary 2.2.9. *$\mathcal{MA}(\kappa)$ implies that the cofinality of 2^ω is bigger than κ (in particular, $\mathcal{MA}$ implies 2^ω is regular).*

Proof. Assume 2^ω has cofinality κ. By $\mathcal{MA}(\kappa)$, the same holds for 2^κ. This contradicts König's Lemma. □

Another application in the same style:

Exercise 2.2.10. *$\mathcal{MA}(\kappa)$ implies that no non-principal ultrafilter on ω can be generated by κ elements.*

2.3. **Applications to topology.** Recall that a subset of a topological set is called *meagre* (or *first category*) if it is the union of countably many nowhere-dense sets (nowhere-dense: the interior of the closure is empty). Intuitively, such sets are small.

Proposition 2.3.1. *Assume $\mathcal{MA}(\kappa)$, and let X be a second-countable topological space such that any open set contains a non-empty proper open subset (for example, X is Hausdorff*[1]*). Then the union of κ many meagre sets is meagre.*

We note that a space is called a *Baire space* if any meagre set in it has empty interior. Thus the Proposition says that in a Baire space as above, the union of κ many closed nowhere dense subsets has empty interior. Since $\mathbb{R}$ satisfies these assumptions (by the Baire category Theorem), it implies $\kappa < 2^\omega$.

End of lecture 13, Sep 21

Proof. We may equivalently prove that given κ open dense sets U_α, there are countably many open dense sets V_n with $\cap_n V_n \subseteq \cap_\alpha U_\alpha$.

The assumptions imply that X has a countable basis B_i with no minimal elements (with respect to inclusion). We would like to find a set $d \subseteq \omega$ such that setting $V_n = \cup_{i \in d \setminus n} B_i$ satisfies the requirements.

The statement that V_n is dense means that for any j there is some $i \in d \setminus n$ with $B_i \cap B_j \neq 0$. In other words, the intersection of d with the set $c_j = \{i \in \omega | B_i \cap B_j \neq 0\}$ is infinite.

To satisfy the inclusion, it is sufficient that for each α there is some n such that $V_n \subseteq U_\alpha$, i.e., such that $B_i \subseteq U_\alpha$ for all $i \in d \setminus n$. In other words, d has finite intersection with $a_\alpha = \{i \in \omega | B_i \not\subseteq U_\alpha\}$.

We wish to apply Theorem 2.2.7 to find such a d. We thus need to show that for every n and every finite collection α_i, there are infinitely many B_k contained in $\cap_i U_{\alpha_i}$ that intersect B_n. Since a finite intersection of open dense sets is again such, it is enough that there are infinitely many B_k in $B_n \cap U$, where U is an open dense set. This is precisely the condition we have on the basis. ∎

2.3.2. *Products of ccc spaces.* Next, we want to ask: Is the product of ccc spaces ccc? We start with the product of two spaces. More generally, is the product of two ccc posets (with the product order) ccc? We attack these questions by introducing a stronger notion that is easily seen to be closed under products.

Definition 2.3.3. A poset $\mathbb{P}$ is *strongly ccc* if any uncountable subset of $\mathbb{P}$ has an uncountable subset of pairwise compatible elements.

It is clear that strongly ccc implies ccc. We note that the ccc sets we used in most of our applications of $\mathcal{MA}$ were in fact strongly so. The exception is a Suslin tree (exercise). We also have:

Exercise 2.3.4. *The product of two strongly ccc posets is strongly ccc*

Theorem 2.3.5. *Assume $\mathcal{MA}(\omega_1)$. Then any ccc poset $\mathbb{P}$ is strongly ccc*

Proof. Let $W = \{w_\alpha | \alpha < \omega_1\} \subseteq \mathbb{P}$. Assume first that any element of $\mathbb{P}$ is compatible with uncountably many elements of W. Let $D_\alpha = \{p \in \mathbb{P} | p \leq w_\beta, \beta > \alpha\}$. The assumption on $\mathbb{P}$ says that each D_α is dense (if q is any element of $\mathbb{P}$, it is compatible

[1] For this condition to be satisfied, we also need to assume X to be perfect. However, isolated points can never belong to a meagre set, so we can ignore them

with some w_β for $\beta > \alpha$). A filter intersecting all of them will intersect W at an uncountable set.

In general, there is an element p_0 such that $\{p \in \mathbb{P} | p \leq p_0\}$ satisfies the assumption (exercise; assume not, and construct an uncountable anti-chain). □

We now pass to the infinite case. It is easy to see that the product of infinitely many ccc (or even finite) orders need not have ccc. But we may find an interesting ccc subset.

Let $\mathbb{P}_a$ be a collection of posets, indexed by $a \in A$. For an element $f \in \prod_a \mathbb{P}_a$, we define the *support* $|f|$ of f to be the set elements $a \in A$ for which $f(a)$ is not a biggest element in $\mathbb{P}_a$. Define $\mathbb{P} = \sum_{a \in A} \mathbb{P}_a$ to be the set of elements with finite support, with order coming from the product.

We note that for finite A, the product and the sum coincide. Also, f and g are compatible in $\mathbb{P}$ if and only if $f(a)$ and $g(a)$ are compatible for every $a \in |f| \cap |g|$. In particular, if the supports are disjoint, they are automatically compatible.

Proposition 2.3.6. *Let $\mathbb{P}_a$ be a poset for each $a \in A$. Then $\mathbb{P} = \sum_{a \in A} \mathbb{P}_a$ has the ccc if and only if $\sum_{a \in A_0} \mathbb{P}_a$ has it for all finite $A_0 \subseteq A$. In particular, $\mathcal{MA}(\omega_1)$ implies that the sum of ccc posets is ccc.*

Proof. Assume that $X \subseteq \mathbb{P}$ is an uncountable anti-chain, and consider the support map $|-| : X \to \mathcal{P}A$. This is a map from an uncountable set to the set of finite subsets of A, so it follows from the Lemma below that there is a finite set A_0 and an uncountable subset Y of X with the property that for distinct $f, g \in Y$, $|f| \cap |g| = A_0$. It follows that the restriction to A_0 gives an uncountable anti-chain in $\sum_{a \in A_0} \mathbb{P}_a$. □

End of lecture 14, Sep 23

Lemma 2.3.7. *Any uncountable collection $\mathscr{A}$ of finite subsets of a set X contains an uncountable collection $\mathscr{B}$ such that for a finite set r, $s \cap t = r$ for any distinct $s, t \in \mathscr{B}$.*

A collection $\mathscr{B}$ as in the Lemma is called a Δ-*system*, and the Lemma is called the "Δ-system Lemma". We note that instead of collections, we may allow repetitions, i.e., replace them by functions from an uncountable domain: Given such a function, if its range is uncountable, the Lemma applies directly. Otherwise, there is an uncountable fibre, and we are done.

Proof. We may assume all sets in $\mathscr{A}$ have the same size n. We prove by induction on n. For $x \in X$, let $\mathscr{A}_x = \{A \in \mathscr{A} | x \in A\}$. If some $\mathscr{A}_x$ is uncountable, remove x from all elements of $\mathscr{A}_x$ and use induction. Otherwise, we get that any countable subset B of X intersect only countably many sets in $\mathscr{A}$ (namely, those in $\cup_{x \in B} \mathscr{A}_x$). Build a sequence s_α ($\alpha < \omega_1$), of sets in $\mathscr{A}$ such that s_α is disjoint from $\cup_{\beta < \alpha} s_\beta$. Then the s_α are pairwise disjoint. □

This concludes the proof of Proposition 2.3.6. Here is an interesting special case, that will be used later in the context of forcing:

Corollary 2.3.8. *Let P be the set of finite partial functions from I to J, ordered by reverse inclusion. If J is countable, then P has the ccc.*

Proof. Add an element $*$ to J, and order it by $a \leq b$ if $a = b$ or $b = *$. Since J is countable, each finite sum has the ccc. □

And here is the application to topological spaces.

Corollary 2.3.9. *Let X_a be a collection of topological spaces for $a \in A$. If any finite product of them has the ccc, then the full product has it. In particular, $\mathcal{MA}(\omega_1)$ implies that if each X_a is ccc, then so is the product.*

Proof. Let P_a be the collection of open sets in X_a (ordered by inclusion), and recall that the elements of $\sum_a P_a$ correspond to a basis for the topology on $\prod_a X_a$. □

2.3.10. *Hausdorff compact ccc spaces.* In most applications of Martin's axiom above, the poset in question was anti-symmetric. Also, it was of cardinality κ when we were using $\mathcal{MA}(\kappa)$. How special is that?

Theorem 2.3.11. *The following are equivalent*

(1) $\mathcal{MA}(\kappa)$
(2) $\mathcal{MA}(\kappa)$ *restricted to orders of cardinality* $\leq \kappa$.
(3) $\mathcal{MA}(\kappa)$ *restricted to complete boolean algebras.*
(4) *If X is a compact ccc topological space, the intersection of κ open dense subsets is non-empty (and dense).*

We recall that a boolean algebra is *complete* if it has arbitrary suprema and infima.

Some implications are straightforward:

Proof of (1) $\implies$ (4). If U is a dense open subset, the set D of open subsets of U is dense (in the poset of open sets). Given a collection U_α of dense open subsets, $\mathcal{MA}(\kappa)$ gives an ultrafilter that intersects each D_α, hence contains each U_α. Since X is compact, the ultrafilter has the form $\{U | x \in U\}$ for some x. □

(2) $\implies$ (1) follows from downwards Löwenheim–Skolem. More generally, we have: $\mathcal{MA}(\kappa)$ for an elementary class of orders (e.g., boolean algebras) is equivalent to the same for orders of size κ.

To prove (4) $\implies$ (3), we need to construct a compact space from each boolean algebra. This is easy: the Stone space is such.

Proof of (4) $\implies$ (3). Given a complete boolean algebra $\mathbb{B}$, let X be its stone space. X clearly has the ccc. For each D_α, let U_α be the sup, viewed as an open set of X. Since D_α is dense, so is U_α. A point of X is an ultrafilter as required. □

End of lecture 15, Sep 26

To prove (3) $\implies$ (2), we need to find a way to produce a boolean algebra from an arbitrary order. More precisely, we have the following.

Proposition 2.3.12. *For any poset $\mathbb{P}$, there is a complete boolean algebra $\mathbb{B} = \mathbb{B}(\mathbb{P})$ and a map $i : \mathbb{P} \to \mathbb{B}$ (of posets) such that*

(1) $i(\mathbb{P})$ *is dense in* $\mathbb{B}$.
(2) $x, y \in \mathbb{P}$ *are compatible if and only if* $i(x)$ *and* $i(y)$ *are compatible.*

The boolean algebra is unique, up to unique isomorphism over $\mathbb{P}$.

We first construct, for each topological space the boolean algebra of regular open sets.

Exercise 2.3.13. *Let X be a topological space. For a subset $Y \subseteq X$, let $\neg Y$ be the complement of the closure of Y. Y is called* regular open *if $\neg\neg Y = Y$. Show that the set of regular open subsets forms a complete boolean algebra, with* inf *given by intersection, and $\sup_\alpha Y_\alpha = \neg\neg(\cup_\alpha Y_\alpha)$.*

Proof of Proposition 2.3.12. Consider the topology on $\mathbb{P}$ given by Remark 2.1.5, let $\mathbb{B}$ be the boolean algebra of regular open sets, and let $i(x) = \neg\neg U_x$. Everything is obvious. □

Proof of Theorem 2.3.11, (3) $\implies$ (1). Let $\mathbb{P}$ be a ccc poset of size κ, and let $i : \mathbb{P} \to \mathbb{B}$ be the map to the corresponding boolean algebra. Clearly $\mathbb{B}$ is ccc. Given dense subsets D_α of $\mathbb{P}$, the sets $i(D_\alpha)$ are dense in $\mathbb{B}$. Hence we have a filter $\mathscr{F}_0$ in $\mathbb{B}$ intersecting each $i(D_\alpha)$. Hence $\mathscr{F} = i^{-1}(\mathscr{F}_0)$ intersects each D_α. The problem is that $\mathscr{F}$ need not be pairwise compatible. We note, however, that it is pairwise compatible in $\mathbb{P}$.

To solve this problem, we add, for each $x, y \in \mathbb{P}$, a dense set

$$D_{xy} = \{z \in \mathbb{P} | (z \leq x \wedge z \leq y) \vee z \perp x \vee z \perp y\}$$

to the collection of sets. It is easy to see that it is dense, and for $x, y \in \mathscr{F}$, this set produces a $z \in \mathscr{F}$ with $z \leq x, y$ (the other options cannot occur, since $\mathscr{F}$ consists of compatible elements). □

3. Forcing

3.1. **Absoluteness.** We consider some model V of set theory, and a countable sub-model M. We would to go as easily as possible between V and M, so we would like some formulas to preserve their truth values (on elements of M). For example, it could be nice if the empty set of M is also the empty set of V. We could take M to be an elementary sub-model, but this will prove counter-productive if we wish M to satisfy different axioms.

Definition 3.1.1. Let $M \subseteq N$ be two structures (in some language). We say that a formula $\phi(x)$ is *absolute* for this inclusion if $\phi(M) = \phi(N) \cap M$. We say that ϕ is absolute for a class of structures if it is absolute for any map in the class.

Obviously, the notion of absoluteness depends only on the definable set determined by ϕ (with respect to the common theory of M and N).

Example 3.1.2. *M is an elementary substructure of N precisely if every formula is absolute for the inclusion.*

Example 3.1.3. *Any quantifier free formula is absolute*

Example 3.1.4. *If ϕ is existential, then $\phi(M) \subseteq \phi(N) \cap M$. Likewise, if ϕ is universal, then $\phi(M) \supseteq \phi(N) \cap M$. Hence, if a formula is both existential and universal, then it is absolute.*

End of lecture 16, Sep 28

We see that the collection of absolute formulas (with respect to $\mathscr{A}$) is a boolean sub-algebra. Now, we restrict to models of set theory. Given an inclusion $M \subseteq N$, recall that M is *transitive* if $x \in^N y \in M$ implies that $x \in M$ (i.e., M is closed downwards).

Proposition 3.1.5. *Assume that $\psi(x, y, z)$ is absolute for transitive $M \subseteq N$. Then so is $\exists z \in y(\psi(x, y, z))$*

Proof. Assume $x, y \in M$ satisfy the formula. Since M is transitive, any witness z to that is in M (in other words, the transitive inclusions are those that remain substructures after adding Skolem functions for such formulas). □

Note that the negation of a formula as in the proposition has the form $\forall z \in y\psi(x, y, z)$. We call such formulas *boundedly quantified*.

Definition 3.1.6. Δ_0 is the smallest boolean algebra of formulas containing the quantifier free formula and closed under bounded quantification.

Corollary 3.1.7. *All the formulas in* Δ_0 *are absolute for transitive inclusions.*

Corollary 3.1.8. *Plenty of formulas are absolute for transitive inclusions of models of* $\mathcal{ZF} - \mathcal{P}$*. For example:*

(1) $x \subseteq y$
(2) *Boolean operations on sets, pair formation, Cartesian products, etc.*
(3) x *is transitive*
(4) x *is an ordinal,* x *is a limit ordinal,* x *is finite*
(5) *The successor function, any finite subset,* ω
(6) R *is a relation, or a function, or an injective function, etc.*
(7) R *is a well order.*
(8) *Anything defined by recursion (using absolute functions). In particular, ordinal arithmetic, etc.*

Proof. Exercise □

Example 3.1.9. *The following notions are not absolute (even for transitive inclusions of models of* $\mathcal{ZFC}$*)*

(1) *The power set functions (i.e., "*y *is the power-set of* x*")*
(2) *Cardinality*

Note, however, that for transitive inclusions, $(\mathcal{P}x)^M = (\mathcal{P}x)^N \cap M$ *(this is just rephrasing the absoluteness of* $y \subseteq x$*).*

The corollary above provides motivation for considering transitive sub-models. What about existence? We have a strong existence result, which we formulate a bit more generally. We say that a binary relation $R(x, y)$ is *set-like* if for any y there is z such that $R(x, y) \iff x \in z$ (in other words, the fibres are sets). We say that R is *extensional* if all fibres are different.

Theorem 3.1.10 (Mostowski collapse)**.** *If* R *is a definable relation on a definable set* X *in a model* V*, and* R *is set-like, extensional and well-founded (in* V*), then* (X, R) *is uniquely isomorphic to a unique transitive sub-structure* M *of* V*. In particular, any sub-model is uniquely isomorphic to a unique transitive one.*

Proof. Uniqueness is clear, since any isomorphism between transitive sub-structures is the identity (by induction).

Given (X, R), define $G : X \to V$ by recursion, via $G(x) = \{G(y)|R(y, x)\}$. This is well defined since the collection of such ys is a set, and R is well founded. Since R is extensional, G is injective (by induction). Also, it is obvious that the image is transitive, and that G is an isomorphism. □

3.2. **Generic filters.** We fix a "big" model V of $\mathcal{ZFC}$. By a *countable transitive model* (ctm) we mean a countable transitive sub-structure of V, satisfying a decent portion of $\mathcal{ZFC}$. Starting with a ctm M, a partial order $\mathbb{P}$ in M, and a generic filter $\mathcal{G}$ for $\mathbb{P}$ (as defined below), forcing produces a ctm extension $M[\mathcal{G}]$ of M including $\mathcal{G}$ (as an element). The construction can be viewed as a generalisation of ultra-products.

By a partial order (poset), we will mean the same as before (Definition 2.1.2), except we assume to have a singled-out biggest element $\mathbf{1}$ (it seems safe to assume anti-symmetry, in which case $\mathbf{1}$ is unique).

Definition 3.2.1. Let M be a ctm, and let $\mathbb{P}$ be a poset in M. A *generic filter* (for $\mathbb{P}$, over M) is a filter of $\mathbb{P}$ (in V), which is $\mathscr{D}$ generic, where $\mathscr{D}$ is the collection of dense subsets of $\mathbb{P}$ that are in M.

End of lecture 17, Sep 30

We note that $\mathbb{P}$ is countable, and therefore ccc. Also, the collection $\mathscr{D}$ in the definition is countable, so by $\mathcal{MA}(\omega)$ (Proposition 2.1.6), generic filters exist. However, except for trivial cases, it will not be an element of M:

Definition 3.2.2. An *atom* of $\mathbb{P}$ is an element $p \in \mathbb{P}$ such that no $q, r \leq p$ are incompatible. $\mathbb{P}$ is called *atomless* if it contains no atom.

Lemma 3.2.3. *Assume that $\mathbb{P}$ is atomless. Then any generic $\mathcal{G}$ is not in M.*

We note that if p is an atom in $\mathbb{P}$, then the set of all elements compatible with p is a filter. This filter will intersect all dense subsets (not just those in M), and forcing with it will give nothing new.

Proof. Assume that $\mathcal{G}$ is in M, and let $D = \mathbb{P} \setminus \mathcal{G}$ (a set in M). If $p \in \mathbb{P}$, there are, by assumption, incompatible elements q and r below it. Since they are incompatible, at least one is not in $\mathcal{G}$, hence in D. This shows that D is a dense subset in M not intersecting $\mathcal{G}$, contradicting genericity. □

End of lecture 18, Oct 3

3.3. **Names and the construction of $M[\mathcal{G}]$.** Fix a partial order $\mathbb{P}$ in a ctm M. We will define structures $M[\mathcal{G}]$, one for each filter $\mathcal{G}$. The structure is constructed as an assignment to a set of constants, called *names*, which we now define.

Definition 3.3.1. The set of $\mathbb{P}$-names is defined by recursion as the smallest collection C of sets of pairs, such that $0 \in C$, and such that any set of pairs (n, p) with $n \in C$ and $p \in \mathbb{P}$ is in C. We denote by $M^{\mathbb{P}}$ the collection of $\mathbb{P}$-names that are elements of M.

Example 3.3.2. *The sets* 0 *and* $\{(0, p), (0, q)\}$ *are names (where* $p, q \in \mathbb{P}$*)*

We think of elements of $M^{\mathbb{P}}$ as names for prospective elements of a structure $M[\mathcal{G}]$. In particular, we expect to have names that name the elements of M and the element $\mathcal{G}$. We can describe them explicitly (recall that $\mathbf{1}$ is a biggest element of $\mathbb{P}$).

Definition 3.3.3. The name $\check{m}$, for $m \in M$ is defined recursively by $\check{m} = \{(\check{n}, \mathbf{1}) | n \in m\}$. We define τ to be the name $\{(\check{p}, p) | p \in \mathbb{P}\}$.

Intuitively, a membership $(n, p) \in c$ when n and c are names will signify that we want the element named by n to be a member of the element named by c, provided that "p is true". We make this definition precise via the following definition.

Definition 3.3.4. We let $\mathcal{T} = \mathcal{T}_{M^{\mathbb{P}}}$ be the following first order theory. The language of $\mathcal{T}$ is the language of set theory, expanded by a constant for each $\mathbb{P}$-name. The theory includes the axiom of extensionality, the atomic diagram of M (when $\check{m}$ is interpreted as m), and the statement that τ is a filter in $\mathbb{P}$.

In addition, for every name σ, $\mathcal{T}$ includes the sentence $\check{p} \in \tau \to \theta \in \sigma$ for every $(\theta, p) \in \sigma$.

We note that by assumption, $\mathbb{P}$ is an element of M, and so has a name $\check{\mathbb{P}}$ that corresponds to a constant symbol of $\mathcal{T}$. Likewise, the order on $\mathbb{P}$ is given by another constant, so the statement that τ is a filter is expressible in the language of $\mathcal{T}$.

So far, all definitions were independent of any filters. Given a filter $\mathcal{G}$, we are supposed to get an actual structure $M[\mathcal{G}]$. We do that in the next definition.

Definition 3.3.5. Let $\mathcal{G}$ be a filter in $\mathbb{P}$. We define, for names σ (by recursion),

$$\mathrm{val}_{\mathcal{G}}(\sigma) = \sigma_{\mathcal{G}} = \{\theta_{\mathcal{G}} | (\theta, p) \in \sigma \quad \text{for some } p \in \mathcal{G}\ \} \tag{3}$$

We define $M[\mathcal{G}] = \{\sigma_{\mathcal{G}} | \sigma \in M^{\mathbb{P}}\}$.

We let $\mathcal{T}^{\mathcal{G}}$ be the extension of $\mathcal{T}$ by the additional sentences $\check{p} \in \tau$ for $p \in \mathcal{G}$.

We make a few easy observations.

Proposition 3.3.6. *Let $\mathcal{G}$ be a filter.*

1. *For all $m \in M$, $\check{m}_{\mathcal{G}} = m$, and $\tau_{\mathcal{G}} = \mathcal{G}$. Hence $M[\mathcal{G}]$ is a transitive subset containing M and $\mathcal{G}$.*
2. *$M[\mathcal{G}]$ is a model of $\mathcal{T}^{\mathcal{G}}$.*
3. *For any name σ, $\sigma_{\mathcal{G}} = \{\theta_{\mathcal{G}} | \mathcal{T}^{G} \vdash \theta \in \sigma\}$.*

Proof.
1. By induction (Exercise)
2. We view $M[\mathcal{G}]$ as a structure via $\mathrm{val}_{\mathcal{G}}$. $M[\mathcal{G}]$ is extensional since it is a transitive substructure. It follows from the previous part that τ is interpreted as $\mathcal{G}$, and that the atomic diagram of M is satisfied. By assumption, $\mathcal{G}$ is a filter. It only remains to check that given $(\theta, p) \in \sigma$, the sentence $\check{p} \in \tau \to \theta \in \sigma$ holds in $M[\mathcal{G}]$, i.e., that if $p \in \mathcal{G}$ then $\theta_{\mathcal{G}} \in \sigma_{\mathcal{G}}$. This is just the definition.
3. Any element of $\sigma_{\mathcal{G}}$ has the form $\theta_{\mathcal{G}}$ where $(\theta, p) \in \sigma$ for some $p \in \mathcal{G}$. Since then $\mathcal{T}^{\mathcal{G}}$ contains the sentences $\check{p} \in \tau \to \theta \in \sigma$ and $\check{p} \in \tau$, hence proves that $\theta \in \sigma$, so the rhs contains $\theta_{\mathcal{G}}$. Conversely, if $\mathcal{T}^{\mathcal{G}} \vdash \theta \in \sigma$, then, by the previous part, $\theta_{\mathcal{G}} \in \sigma_{\mathcal{G}}$. □

Corollary 3.3.7. *Let θ, σ be names, and let $\mathcal{G}$ be a filter such that $\theta_{\mathcal{G}} \in \sigma_{\mathcal{G}}$. Then there is $p \in \mathcal{G}$ such that $\mathcal{T} \vdash \check{p} \in \tau \to \theta \in \sigma$.*

Proof. We know that $\mathcal{T}^{\mathcal{G}} \vdash \theta \in \sigma$. By compactness, a finite number of statements $\check{q} \in \tau$ imply (with respect to $\mathcal{T}$) $\theta \in \sigma$. Since all q in these statements are in $\mathcal{G}$, and $\mathcal{G}$ is a filter, there is an element p in $\mathcal{G}$ below all of them. This element satisfies the claim. □

As mentioned above, we may view this construction as a generalisation of ultra-products:

Exercise 3.3.8. *Let A be an infinite set, and let $\mathbb{P}$ be the poset of infinite subsets of A.*

1. *Show that any generic filter in $\mathbb{P}$ is an ultra-filter (i.e., maximal)*
2. *Show that there is a map i from the set of names (in M) to the product M^A, such that for any generic filter $\mathcal{G}$ in $\mathbb{P}$, the map $\mathrm{val}_{\mathcal{G}}$ factors through i and the map from the product to the ultra-product.*

Corollary 3.3.9. *For any $p, q \in \mathbb{P}$, $p \leq q$ if and only if $\mathcal{T} \vdash \check{p} \in \tau \to \check{q} \in \tau$.*

Proof. Assume $p \leq q$. Then any filter that contains p also contains q. For the converse, consider $M[\mathcal{G}]$, where $\mathcal{G}$ is the filter generated by p. □

Hence, the order on $\mathbb{P}$ can be recovered from the order of implication with respect to $\mathcal{T}$.

End of lecture 19, Oct 10

We mention other properties of $M[\mathcal{G}]$ that are most easily seen explicitly from the construction of names. Given a name σ, we denote by $\mathrm{dom}(\sigma)$ the set of names θ such that $(\theta, p) \in \sigma$ for some p.

Proposition 3.3.10. *Let $\mathcal{G}$ be a filter.*

(1) *M and $M[\mathcal{G}]$ have the same ordinals.*
(2) *If $\mathcal{G} \in M$, then $M[\mathcal{G}] = M$*
(3) *If N is a transitive substructure of V containing M and including $\mathcal{G}$, then $M[\mathcal{G}] \subseteq N$.*
(4) *$M[\mathcal{G}]$ satisfies the axioms Extensionality, Foundation, Pairing, Union, Infinity and the axiom of choice.*

Proof. (1) Since $M \subseteq M[\mathcal{G}]$, and both are transitive, the ordinals of M are the ordinals of $M[\mathcal{G}]$ that are in M. If $\alpha = \sigma_{\mathcal{G}}$ is an ordinal of $M[\mathcal{G}]$, there is a subset of σ in M of order type α. Hence $\alpha \in M$.

(2) Exercise

(3) The function $\mathrm{val}_{\mathcal{G}}$ is absolute (exercise), so $M[\mathcal{G}] = \{\sigma_{\mathcal{G}} \in N[\mathcal{G}] | \sigma \in M\}$. But since $\mathcal{G} \in N$, $N[\mathcal{G}] = N$.

(4) Extensionality and Foundation are satisfied in any transitive substructure. If $\sigma_{\mathcal{G}}, \theta_{\mathcal{G}} \in M[\mathcal{G}]$, $\{(\sigma, \mathbf{1}), (\theta, 1)\}$ is a name for the pair. Since ω is absolute, infinity holds as well.

For union, given $\sigma_{\mathcal{G}} \in M[\mathcal{G}]$, let $\pi = \cup\{\mathrm{dom}(\theta) | \theta \in \sigma\}$. It is clear that π is a name and that $\cup\sigma_{\mathcal{G}} \subseteq \pi_{\mathcal{G}}$.

For the axiom of choice, given σ, let $f : \alpha \to \mathrm{dom}(\sigma)$ be a well ordering of its domain (in M). Then $\{(f, \mathbf{1})\}$ is a name for a well ordering of a set containing $\sigma_{\mathcal{G}}$ (this is a bit sloppy). □

3.3.11. *Definability of $\mathcal{T}$.* Let $\mathcal{L}$ be a first order language, for simplicity one-sorted and without function symbols. We say that the signature of $\mathcal{L}$ is definable in M, if the sets C and R_n of constant and n-ary relation symbols are definable subsets of M. In this case, the set of words on the logical alphabet (i.e., finite sequences of constant symbols, relation symbols, variables and logical symbols) is also definable, as well as the subset $\mathcal{L}$ consisting of the language, i.e., the well formed formulas. Likewise, the subset of $\mathcal{L}$ consisting of quantifier free formulas, or formulas where a particular variable occurs free, etc. are definable subsets of $\mathcal{L}$. All of this follows from the basic definitions of first order logic, together with the existence Theorem for definable sets in $\mathcal{ZFC}$ defined by recursion (Kunen [4, § I.9]), and the fact that we are working in a transitive model, so notions like "finite sequence", and more generally, definitions by recursion, are unambiguous (Corollary 3.1.8). We also note that, for similar reasons, the relation $\phi \vdash \psi$ on sentences in $\mathcal{L}$ is definable.

We say that a first order theory $\mathcal{T}_0$ is definable if it has a definable signature, so that $\mathcal{T}_0$ is a definable subset the corresponding language $\mathcal{L}$. We note that in this context, by a theory we simply mean a set of sentences. However, it follows from the remarks above that the set of formulas ϕ such that $\mathcal{T}_0 \vdash \phi$ is definable (by the formula $\exists\psi \in \mathcal{T}_0^*(\psi \vdash \phi)$, where $\mathcal{T}_0^*$ is the (definable) closure of $\mathcal{T}_0$ under finite conjunctions).

Proposition 3.3.12. *The theory $\mathcal{T}$ defined above is definable. In particular, given names θ, σ, the set of $p \in \mathbb{P}$ such that $\mathcal{T} \vdash \check{p} \in \tau \to \theta \in \sigma$ is definable (and is therefore an element of M).*

Proof. By definition, we need to show that the sets of constants and relation symbols are definable, and that $\mathcal{T}$ is a definable subset of the language. The constants are the names, and their definability again follows from definition by recursion. The set of relation symbols consists of one binary relation, and so is definable.

The map $m \mapsto \check{m}$ (from M to the definable set of names) is definable, again by recursion. The diagram is the image of the membership relation on M under this map (more precisely, under the map $(m, n) \mapsto R(\check{m}, \check{n})$, where R is the relation of $\mathcal{T}$). The statement that τ is a filter is given by one sentence, and is therefore definable. The rest of the sentences are again visibly a definable function of the names and elements of $\mathbb{P}$. □

We note that in contrast, $\mathcal{T}^{\mathcal{G}}$ is definable if and only if $\mathcal{G}$ is in M.

Corollary 3.3.13. *For any filter $\mathcal{G}$, $M[\mathcal{G}]$ satisfies the Power Set axiom*

Proof. Given σ, let $\theta = \mathcal{P}(\mathrm{dom}(\sigma) \times \mathbb{P}) \times \{\mathbf{1}\}$. Hence, a typical element of $\theta_{\mathcal{G}}$ has the form $\pi_{\mathcal{G}}$, where π is a name whose domain is contained in the domain of σ. Given $\mu_{\mathcal{G}} \subseteq \sigma_{\mathcal{G}}$, let $\pi = \{(\rho, p) | \rho \in \mathrm{dom}(\sigma), \mathcal{T} \vdash \check{p} \in \tau \to \rho \in \mu\}$. By Proposition 3.3.12, the condition defining π is definable, so π is a name in M, whose domain is contained $\mathrm{dom}(\sigma)$. Hence, $\pi_{\mathcal{G}} \in \theta_{\mathcal{G}}$. We claim that $\pi_{\mathcal{G}} = \mu_{\mathcal{G}}$: if $\rho_{\mathcal{G}} \in \mu_{\mathcal{G}}$, we may assume that $\rho \in \mathrm{dom}(\sigma)$, since $\mu_{\mathcal{G}} \subseteq \sigma_{\mathcal{G}}$, so $\mu_{\mathcal{G}} \subseteq \pi_{\mathcal{G}}$ by Proposition 3.3.6. The converse is obvious. □

3.4. **Forcing.** We now come to the main definitions and results on forcing. Though we will use the theory $\mathcal{T}$ discussed above, the details of its construction are no longer important. The main properties of $\mathcal{T}$ that we will use are:

(1) $\mathcal{T}$ is an expansion by constants of the atomic diagram of M. There is a distinguished constant τ, and $\mathcal{T}$ implies that τ is a filter of $\mathbb{P}$.
(2) Given a filter $\mathcal{G}$ of $\mathbb{P}$, it is possible to expand the assignment $\tau \mapsto \mathcal{G}$ to a canonical model $M[\mathcal{G}]$ of $\mathcal{T}$, such that each element of $M[\mathcal{G}]$ is named by a constant symbol of $\mathcal{T}$ (Proposition 3.3.6).
(3) $\mathcal{T}$ is definable in M (Proposition 3.3.12).
(4) Completeness: If c, d are constant symbols such that $M[\mathcal{G}] \models c \in d$, then for some $p \in \mathcal{G}$, $\mathcal{T} \vdash p \in \tau \to c \in d$ (Corollary 3.3.7).

Definitions and results in this subsection should be valid for any theory $\mathcal{T}$ satisfying these assumptions, unless mentioned otherwise. In particular, we will identify from now on $m \in M$ with the constant $\check{m}$ of $\mathcal{T}$ (as we have done in the statement of completeness).

We are interested to know what statements are true in a structure $M[\mathcal{G}]$.

Definition 3.4.1. An element $p \in \mathbb{P}$ *forces* a sentence ϕ of $\mathcal{T}$ (denoted $p \Vdash \phi$) if ϕ is true in $M[\mathcal{G}]$ whenever $\mathcal{G}$ is a generic filter containing p.

The statement $p \Vdash \phi$ can be thought of as the forcing analogue of "ϕ is a consequence of $p \in \tau$", i.e., of $p \in \tau \models \phi$ (relative to $\mathcal{T}$). We would like to define the forcing analogue of "$p \in \tau \vdash \phi$", and prove a "completeness Theorem". We are not actually interested in a proof system here. Rather, the important feature for us is that provability is definable.

End of lecture 20, Oct 5

Proposition 3.4.2. *For each sentence ϕ of $\mathcal{T}$ there is a definable subset $p \vdash \phi$ of $\mathbb{P}$ (in M), such that*

(1) *If $p \vdash \phi$ then $p \Vdash \phi$*
(2) *If $\mathcal{G}$ is a generic filter, and $M[\mathcal{G}] \models \phi$, then $p \vdash \phi$ for some $p \in \mathcal{G}$.*

We will need the following result.

Lemma 3.4.3. *Assume that $\mathcal{G}$ is a generic filter, and $E \in M$ is disjoint from $\mathcal{G}$. Then there is some element $p \in \mathcal{G}$ incompatible with all elements of E.*

Proof. We may assume that E is downwards closed. Let D be E together with all elements incompatible with E. If p is not in D, it is compatible with some element of E. Since E is downwards closed, there is some $q \in E$ below p. Hence D is dense, and $\mathcal{G}$ contains an element of D (note that D is in M). Since it cannot be in E, it is incompatible with all elements in E. □

Proof of 3.4.2. By induction on the complexity of ϕ. When ϕ is $\theta \in \sigma$, we define $p \vdash \phi$ by $\mathcal{T} \vdash p \in \tau \rightarrow \phi$. By Corollary 3.3.12, this is definable in M. (1) follows since each $M[\mathcal{G}]$ is a model of $\mathcal{T}$, and (2) holds by Corollary 3.3.7.

We ignore the case $\theta = \sigma$ since, by extensionality, it is definable from the membership relation.

Assuming that the proposition holds for ϕ and ψ, we define $p \vdash (\phi \wedge \psi)$ by $p \vdash \phi \wedge p \vdash \psi$ (note that the first $\wedge$ is in the language of $\mathcal{T}$, but the second is in M). Clearly, this is definable, and it is easy to check that the conditions are satisfied.

Next, assume the proposition for ϕ. We define $p \vdash \neg\phi$ by $\forall q \in \mathbb{P}(q \vdash \phi \rightarrow p \perp q)$. To prove (1), assume p satisfies the formula, and let $\mathcal{G}$ be a generic filter with $p \in \mathcal{G}$. If $M[\mathcal{G}] \models \phi$, then, by induction, $q \vdash \phi$ for some $q \in \mathcal{G}$. Since p satisfies the formula, we get that $p \perp q$. This contradicts the fact that they are in the same filter.

To prove (2), let $\mathcal{G}$ be a generic filter, such that $M[\mathcal{G}] \models \neg\phi$, and consider the set $E = \{q \in \mathbb{P} | q \vdash \phi\}$. This is (by induction) a definable subset of $\mathbb{P}$ (and hence an element of M). If E intersects $\mathcal{G}$, we have a $q \in \mathcal{G}$ that (by induction) forces ϕ, contradicting that $M[\mathcal{G}] \models \neg\phi$. Hence, by Lemma 3.4.3, there is an element p of $\mathcal{G}$ incompatible with all q in E, i.e., $p \vdash \neg\phi$. (Note that this is the only place in the proof where we explicitly use the genericity of $\mathcal{G}$).

Finally, let $\phi = \exists x\phi(x)$. We define $p \vdash \phi$ by $\exists c \in C(p \vdash \phi(c))$, where C is the (definable) set of constant symbols of $\mathcal{T}$. The proof that this condition satisfies the requirement is straightforward, using the fact that every element of $M[\mathcal{G}]$ is represented by a constant. □

Corollary 3.4.4. (1) *The relation $p \Vdash \phi$ is definable.*
(2) *If $\mathcal{G}$ is generic, and $M[\mathcal{G}] \models \phi$, then $p \Vdash \phi$ for some $p \in \mathcal{G}$.*

We will use the following definition.

Definition 3.4.5. Let $p \in \mathbb{P}$. A subset $E \subseteq \mathbb{P}$ is *dense below* p if E has an element below any $q \leq p$.

Hence, by Lemma 3.4.3, if E is in M, dense below $p \in \mathcal{G}$ (where $\mathcal{G}$ is generic), then E intersects $\mathcal{G}$.

Proof of 3.4.4. The second part follows directly from the Proposition. To prove the first part, we claim that $p \Vdash \phi$ is defined by "$\{q \in \mathbb{P} | \exists r \geq q(r \vdash \phi)\}$ is dense below p". By the Proposition, this is definable.

Assume that p satisfies the condition, and let $\mathcal{G}$ be a generic filter containing p. By the remark above, $r \vdash \phi$ for some $r \geq q \in \mathcal{G}$. Hence, by the proposition, $r \Vdash \phi$, so $M[\mathcal{G}] \models \phi$.

Conversely, if $p \Vdash \phi$, and $t \leq p$, let $\mathcal{G}$ be a generic filter containing t. Then $M[\mathcal{G}] \models \phi$, so by the Proposition, $r \vdash \phi$ for some $r \in \mathcal{G}$. Take $q \in \mathcal{G}$ below both r and t. □

End of lecture 21, Oct 7

Finally, we formulate the main result. For convenience, we also summarise the previous results.

Theorem 3.4.6. *Let M be a ctm, $\mathbb{P} \in M$ a partial order. Given a generic filter $\mathcal{G}$ in $\mathbb{P}$, there is a structure $M[\mathcal{G}]$ with the following properties.*

1. *$M[\mathcal{G}]$ is a ctm extending M and including $\mathcal{G}$, and is contained in any other substructure of V with these properties.*
2. *M and $M[\mathcal{G}]$ have the same ordinals*
3. *A sentence ϕ is satisfied in $M[\mathcal{G}]$ if and only if $p \Vdash \phi$ for some $p \in \mathcal{G}$.*
4. *The relation $p \Vdash \phi$ is definable in M (independently of $\mathcal{G}$)*

Proof. It only remains to prove that $M[\mathcal{G}]$ satisfies Comprehension and Replacement. Let σ be a name, $\phi(x, y)$ a formula (in the language of $\mathcal{T}$). Let

$$\theta = \{(\pi, p) | \pi \in \text{dom}(\sigma), p \Vdash \pi \in \sigma \wedge \phi(\pi, \sigma)\} \quad (4)$$

Then $\theta_{\mathcal{G}} = \{x \in \sigma_{\mathcal{G}} | \phi(x, \sigma_{\mathcal{G}})\}$ (the proof is similar to the proof of Corollary 3.3.13). The proof for Replacement is similar. The facts that $M[\mathcal{G}]$ satisfies the rest of $\mathcal{ZFC}$, that it is minimal, and that the ordinals are the same were proved in Proposition 3.3.10 and Corollary 3.3.13. The last two parts are Corollary 3.4.4. □

3.5. **The consistency of $\neg\mathcal{CH}$.** The Forcing Theorem is applied in a manner similar to the application of Martin's axiom, by finding interesting partial orders. For instance, we can produce a new subset of an arbitrary set in the original model. The following is very similar to Theorem 2.2.7, but we prove it again for practice.

Proposition 3.5.1. *Let M be a ctm, and let $A \in M$ be an infinite set. Then there is a ctm extension of M with a new subset of A. More generally, for any ordinal κ there is a ctm with a family of distinct subset A_α indexed by κ.*

End of lecture 22, Oct 12

Proof. Let $\mathbb{P}$ be the poset of finite partial functions from A to 2 (note that this set is absolute), ordered by reverse inclusion, and let $\mathcal{G}$ be a generic filter in $\mathbb{P}$. For any $a \in A$, the subset of $\mathbb{P}$ consisting of elements defined at a is definable and dense. Hence $g = \cup\mathcal{G}$ is a total function from A to 2. If $B \subseteq A$ is in M, then $\{f \in \mathbb{P} | f \perp \chi_B\}$ is definable and dense, so g is not the characteristic function of any existing set. We note that the new set can be explicitly defined in $\mathcal{T}$ by $(n, 1) \in \cup\tau$.

For the more generally, apply the construction with $A \times \kappa$ in place of A. This determines new subset of $A \times \kappa$ whose fibre are subsets of A. To see that they are all distinct, consider, for any $\alpha, \beta \in \kappa$, the set $\{f \in \mathbb{P} | f_\alpha \perp f_\beta\}$. □

It follows that for an arbitrary set κ in M, we found a ctm with an injective function $\kappa \to 2^\omega$. This *does not* prove the consistency of the negation of $\mathcal{CH}$, however: κ might become countable in the new model.

Hence we need to prove that when we pass to $M[\mathcal{G}]$, the notion of "cardinal" does not change. It turns out that it is more convenient to work with a stronger property.

Definition 3.5.2. $\mathbb{P}$ *preserves cofinalities* if for any generic filter $\mathcal{G}$, the cofinality of of any limit ordinal is the same in M and in $M[\mathcal{G}]$.

Lemma 3.5.3. *If $\mathbb{P}$ preserves cofinalities, then it preserves cardinals*

Proof. The lemma is obvious for regular cardinals. Since a singular cardinal is a limit of regular cardinals, it is also true for them. □

Any poset in M is countable, and therefore ccc, as a member of V. However, M itself need not think that $\mathbb{P}$ is ccc. We will prove:

Theorem 3.5.4. *Assume that $\mathbb{P}$ is ccc according to M. Then $\mathbb{P}$ preserves cofinalities (and thus cardinalities).*

We need two more lemmas:

Lemma 3.5.5. *If any uncountable regular cardinal in M is a regular ordinal in each $M[\mathcal{G}]$, then $\mathbb{P}$ preserves cofinalities.*

Proof. Let κ be the cofinality of α in M. Then there is a strictly increasing $f : \kappa \to \alpha$, with cofinal image. Since κ is a cofinality, it is regular, so by assumption, its cofinality in $M[\mathcal{G}]$ is κ. Since f is still an increasing cofinal map in $M[\mathcal{G}]$, we get that κ is the cofinality of α in $M[\mathcal{G}]$. □

Lemma 3.5.6. *Assume that $\mathbb{P}$ is ccc according to M, and that f is a function in $M[\mathcal{G}]$. Then there is a function F in M, such that $f(a) \in F(a)$ for all a, and each $F(a)$ is countable according to M.*

Proof. There is some $p \in \mathcal{G}$ that forces f to be a function. Let $F(a) = \{b | \exists q \leq p(q \Vdash f(a) = b\})$. By definability of forcing, F is definable in M.

By the axiom of choice, there is, in M, a Skolem function $g : F(a) \to \mathbb{P}$. The image of g is an anti-chain in $\mathbb{P}$, since any two values force different values for $f(a)$. Hence $F(a)$ is countable. □

Proof of 3.5.4. If not, there is an uncountable regular cardinal κ in M that fails to be regular in $M[\mathcal{G}]$ (Lemma 3.5.5). Hence there is a cofinal increasing function $f : \alpha \to \kappa$ in $M[\mathcal{G}]$, with $\alpha < \kappa$. Applying Lemma 3.5.6, we obtain a function $F : \alpha \to \mathcal{P}\kappa$. Since $f(a) \in F(a)$, $\cup F(\alpha)$ is cofinal, and since each $F(a)$ is countable (in M), this set has cardinality α, contradicting the regularity of κ. □

Corollary 3.5.7. *If M is a ctm and $\kappa \in M$ is a cardinal, M has a ctm extension that preserves cardinalities, with $2^\omega \geq \kappa$*

Proof. The poset $\mathbb{P}$ used in Proposition 3.5.1 is ccc (according to M) by Corollary 2.3.8. □

Exercise 3.5.8. *Let $\kappa = 2^{2^\omega}$. Why don't we get a contradiction?*

We would like to obtain a more precise information on what could be the cardinality of 2^ω. In other words, we would like to have an upper bound on the cardinality of the power set. For that purpose, we construct a *name* of small cardinality for the power set.

Definition 3.5.9. Given a name σ, a *nice name* for a subset of σ is a name θ such that $\mathrm{dom}(\theta) \subseteq \mathrm{dom}(\sigma)$, and for each name μ, $\theta_\mu = \{p | (\mu, p) \in \theta\}$ is an anti-chain.

The point is now that any subset can be represented by a nice name (uniformly).

Lemma 3.5.10. *For any names θ, σ, there is a nice name θ' such that $\mathbf{1} \Vdash (\theta \subseteq \sigma \to \theta = \theta')$.*

Proof. We have already seen (in the proof of Comprehension, Theorem 3.4.6) that the same holds for $\theta'' = \{(\mu, p) | \mu \in \mathrm{dom}(\sigma), p \Vdash \mu \in \theta\}$. We let θ' be a subset of θ'' with the same domain, such that each θ'_μ is a maximal anti-chain. We note that θ' is a name by Zorn's Lemma in M.

Assume that for some $\mathcal{G}$ and some $\mu_{\mathcal{G}} \in \theta''_{\mathcal{G}}$, $\mu_{\mathcal{G}} \notin \theta'_{\mathcal{G}}$. We may assume that $\mu \in \mathrm{dom}(\theta'')$. Then $\mathcal{G}$ is disjoint from θ'_μ, so some element of $\mathcal{G}$ is incompatible with all elements in this set, contradicting maximality. □

We may now put an upper bound on 2^κ:

Proposition 3.5.11. *Assume $\mathbb{P}$ has the ccc in M. Let $\kappa = |\mathbb{P}|$, let κ_1 be another cardinal in M, and let $\kappa_2 = \kappa^{\kappa_1}$ in M. Then in $M[\mathcal{G}]$, $2^{\kappa_1} \leq \kappa_2$.*

We note that in $M[\mathcal{G}]$, it is no longer the case (in general) that $\kappa_2 = \kappa^{\kappa_1}$, so the result is not obvious.

Proof. We may represent the power set 2^{κ_1} in $M[\mathcal{G}]$ by a name σ consisting of nice names for subsets of κ_1. Hence, each element of σ has the form $\{(\theta, p) | \theta \in A, p \in C_\theta\}$, where A is a subset of $\mathrm{dom}(\check{\kappa_1})$, and C_θ is an anti-chain, hence countable. Each nice name determines (and is determined by) a function $\theta \mapsto C_\theta$ from κ_1 to the set of anti-chains. Hence the cardinality of σ is bounded by $(\kappa^\omega)^{\kappa_1} = \kappa_2$.

Now, in $M[\mathcal{G}]$, the function $\mathrm{val}_{\mathcal{G}} : \sigma \to 2^{\kappa_1}$ is definable, hence an element of $M[\mathcal{G}]$. So in $M[\mathcal{G}]$, $2^{\kappa_1} \leq |\sigma| \leq \kappa_2$. □

Corollary 3.5.12. *If κ is a cardinal in M such that $\kappa^\omega = \kappa$, then M has a ctm extension with $2^\omega = \kappa$.*

Proof. By the previous Proposition and Corollary 3.5.7. □

End of lecture 23, Oct 14

3.6. **Consistency of $\mathcal{CH}$ and of the Diamond principle.** We now would like to force the Continuum Hypothesis. The first idea is to force a surjective function from ω_1 to $\mathcal{P}\omega$ using the poset $\mathbb{F}n(\omega_1, \mathcal{P}\omega)$ of finite partial functions from ω_1 to $\mathcal{P}\omega$. This idea turns out to work too well:

Exercise 3.6.1. *Show that if $\mathcal{G}$ is a generic filter of $\mathbb{F}n(\omega_1, \mathcal{P}\omega)$, then $g = \cup\mathcal{G}$ restricts to a surjective function from ω to $\mathcal{P}\omega^M$.*

The problem is, thus, that forcing does not preserve the power set. We will define a class of posets that allow us to control this problem.

Definition 3.6.2. The poset $\mathbb{P}$ is *λ-closed* if for any decreasing sequence p_α of length smaller than λ, there is $p \leq p_\alpha$ for all α

Theorem 3.6.3. *Let A and B be sets in M, with $|A| < \lambda$, and let $\mathbb{P}$ be λ-closed (all in the sense of M). Then for any generic $\mathcal{G}$, any function from A to B in $M[\mathcal{G}]$ is in M.*

Proof. We may assume that A is a cardinal. Let K be the set of functions from A to B in M. If the claim is false, there is a name σ and $p_0 \in \mathbb{P}$ such that p_0 forces σ to be a function from $\check{A}$ to $\check{B}$ that is not in $\check{K}$.

We define $p_\alpha \in \mathbb{P}$ and $x_\alpha \in B$ for $\alpha \leq A$ by induction. Assuming p_α and x_β were defined for $\beta < \alpha$, choose $p_{\alpha+1} \leq p_\alpha$ and x_α so that $p_{\alpha+1} \Vdash \sigma(\check{\alpha}) = \check{x_\alpha}$. This is possible since $p_\alpha \le p_0$, and thus forces σ to be a function on A. On limit stages $\alpha < \lambda$, choose any $p_\alpha \le p_\beta$ for $\beta < \alpha$.

The function $\alpha \mapsto x_\alpha$ is in M by transfinite recursion and axiom of choice in M. If $\mathcal{G}$ is any generic filter including p_A, $\sigma(\alpha) = x_\alpha$ in $M[\mathcal{G}]$, a contradiction. □

Corollary 3.6.4. *If $\mathbb{P}$ is λ-closed, it preserves cardinals up to λ, and power sets of sets of cardinality smaller than λ.*

Given a poset $\mathbb{P}$ and a cardinal $\lambda > 0$, it is possible to construct a poset $\mathbb{P}_1$ containing $\mathbb{P}$ and λ-closed, as follows: The elements of $\mathbb{P}_1$ are descending sequences of elements of $\mathbb{P}$, of length smaller than λ. If s and t are two such sequences, $s \le t$ if for any element of t, s contains a smaller element. $\mathbb{P}$ is embedded in $\mathbb{P}_1$ by viewing each element as a sequence of length 1, and it is clear that $\mathbb{P}_1$ is λ-closed. In the case when $\mathbb{P} = \mathbb{F}n(A, B)$, this process has the following explicit description.

Definition 3.6.5. For sets A, B and a cardinal λ of M, the poset $\mathbb{F}n(A, B, \lambda)$ is the set of partial functions from A to B of cardinality smaller than λ.

Hence, $\mathbb{F}n(A, B, \omega) = \mathbb{F}n(A, B)$.

Lemma 3.6.6. *$\mathbb{F}n(A, B, \lambda)$ is $\mathrm{cf}(\lambda)$-closed.*

Proof. Take unions. □

Corollary 3.6.7. *Let $\mathcal{G}$ be a generic filter of $\mathbb{F}n(\omega_1, \mathcal{P}\omega, \omega_1)$. Then $M[\mathcal{G}]$ satisfies $\mathcal{CH}$.*

Proof. The function $g = \cup\mathcal{G} : \omega_1 \to \mathcal{P}\omega$ is onto (exercise), and by Corollary 3.6.4, $\mathcal{P}\omega$ is the same in M and $M[\mathcal{G}]$. □

3.6.8. *Forcing with $\mathbb{F}n(I, J, \kappa)$.* The poset $\mathbb{P} = \mathbb{F}n(I, J, \kappa)$ does not, in general, satisfy the ccc. However, it satisfies the obvious generalisation concerning antichains of larger cardinality. It is then possible to prove analogues of the results above. The proofs are mostly the same, so we omit them, but we summarise the main facts in the case when κ is regular and $1 < |J| \le \kappa$.

(1) $\mathbb{P}$ is κ-closed (Lemma 3.6.6), and therefore preserves cardinalities up to κ, and power sets of sets of smaller cardinalities (Corollary 3.6.4).
(2) If $I = A \times B$, then $\mathbb{P}$ forces a function $A \to J^B$, which is injective if $|B| \ge \kappa$ (same as Proposition 3.5.1), and surjective if $|A| \ge \kappa$ and $|B| < \kappa$ (similar to Corollary 3.6.7, using the previous part). More generally, without the restriction on $|B|$, the map $A \to J^B$ induces a surjective map $A \to J^{B_0}$ for any $B_0 \subseteq B$ with $|B_0| < \kappa$. In particular, setting $A = B = \kappa$ and $J = 2$ we get $2^{<\kappa} = \kappa$.
(3) $\mathbb{P}$ preserves cardinals (and cofinalities) $\ge (2^{<\kappa})^+$. The main point: $\mathbb{P}$ has the $(2^{<\kappa})^+$-cc. The proof, given below, is similar to Corollary 2.3.8, using a stronger version of the Δ-system lemma (Proposition 3.6.9). It follows from this and the first part that if $2^{<\kappa} = \kappa$, then $\mathbb{P}$ preserves all cardinalities (and cofinalities). Hence in this case, we may use $\mathbb{P}$ to force $\lambda \le 2^\kappa$, while preserving cardinalities of power sets below κ (take $A = \lambda, B = \kappa, J = 2$). If we also assume $\lambda > \kappa$ and $\lambda^\kappa = \lambda$, then we get $\lambda = 2^\kappa$, counting nice names as in Corollary 3.5.12.

Using this, it is possible to force $\mathcal{CH}$ while violating the generalised Continuum Hypothesis. See Kunen [4, § VII.6.18]. This will be more thoroughly discussed in the next section.

We now prove the main point missing in the above discussion, namely the (anti-) chain condition for posets of partial functions. Recall that in the case of finite support, the analogous fact (Proposition 2.3.6) was proved using the Δ-system lemma. This lemma has the following generalisation.

Proposition 3.6.9 (Δ-system lemma)**.** *Let $\mathscr{A}$ be a collection of size θ of sets of size κ. Assume that $\theta \geq \omega$ is regular, and that $\lambda^\kappa < \theta$ for all $\lambda < \theta$. Then there is a subset $\mathscr{B} \subseteq \mathscr{A}$ of cardinality θ (a* Δ-system*), and a set R (the* root*), such that $X \cap Y = R$ for all distinct $X, Y \in \mathscr{B}$.*

Corollary 3.6.10. *Let λ be an infinite cardinal, $\theta = (|B|^{<\lambda})^+$. Then $\mathbb{F}n(A, B, \lambda)$ has the θ-cc*

Proof. Assume first that λ is regular, and let C be an anti-chain of size θ. Let C_κ, $\kappa < \lambda$ be the set of elements of C of size κ. Since θ is regular, some C_κ has cardinality θ. Hence we may assume that all elements of C have cardinality κ. Further, $(|B|^{<\lambda})^\kappa = |B|^{<\lambda} < \theta$. Hence the conditions of the Δ-system lemma are sastisfied for the collection $\mathscr{A}$ of domains of elements of C. So we may assume that there is a subset R of A contained in the domain of all elements of C, such that the restriction of all elements of C to R are pairwise distinct. But there are only $|B|^{|R|} \leq |B|^\kappa < \theta$ such functions.

When λ is singular, write it as a limit of regulars. □

Proof of Proposition 3.6.9, sketch. By cardinality arguments as above, we may assume: $\mathscr{A}$ consists of subsets of θ of the same order type ρ, and $\cup\mathscr{A}$ is unbounded in θ. We view each element of $\mathscr{A}$ as an increasing function on ρ.

Since θ is regular and $\rho < \theta$, there is some $\alpha < \rho$ such that $\{x(\alpha) | x \in \mathscr{A}\}$ is unbounded. Let β be the least such. Let $C = \{x(\alpha) | x \in \mathscr{A}, \alpha < \beta\}$. Then C is bounded by some $\alpha < \theta$, and $|C| \leq |\alpha|^{|\beta|} < \theta$. It follows that there is a fixed $x : \beta \to \alpha$ such that θ elements of $\mathscr{A}$ restrict to x. We thus assume all elements of C to be such.

Now choose elements x_μ of C by induction, so that $x_\mu(\beta) > x_\nu(\gamma)$ for all $\nu < \mu$ and all γ. □

End of lecture 24, Oct 24

3.6.11. *The Diamond principle.* We now prove the consistency of the Diamond principle. We will use a generic filter to produce a diamond sequence: let $\mathbb{P} = \mathbb{P}_\diamondsuit$ be the poset of countable sequences A_α, where $A_\alpha \subseteq \alpha$, where $s \leq t$ if s extends t. We note that $\mathbb{P}_\diamondsuit$ is a sub-order of $\mathbb{F}n(\Gamma, 2, \omega_1)$, where Γ is the order relation on ω_1, and this order can be used instead. Both are ω_1-closed. We denote the length of an element p of $\mathbb{P}$ by $|p|$.

Proposition 3.6.12. *Let $\mathcal{G}$ be $\mathbb{P}_\diamondsuit$-generic. Then $\cup\mathcal{G}$ is a diamond sequence in $M[\mathcal{G}]$.*

Proof. Let τ be the name for $\cup\mathcal{G}$, σ the name of a club, θ the name of a subset of ω_1 (note that ω_1 is unambiguous), and $p \in \mathcal{G}$ an element forcing that. We need to show that for some $\alpha \in \sigma$, $\theta \cap \alpha = A_\alpha$. It is sufficient to show that the set of q that force this (for some $\alpha < |q|$) is dense below p. Hence we start with some $p_0 \leq p$, and find q below it forcing the condition.

To find q, we construct by induction a descending sequence $p_i \in \mathbb{P}$, elements $\alpha_i > |p_i|$ and subsets $B_i \subseteq |p_i|$, such that $p_{i+1} \Vdash \alpha_i \in \sigma \wedge \theta \cap |p_i| = \check{B}_i$. Assume that p_i and α_j $(j < i)$ were defined. Let $\beta_i = |p_i|$. Since $p_i < p_0 \leq p$, it forces σ to be unbounded. Hence there are $\alpha_i > \beta_i$ and $t_i < p_i$ with $t_i \Vdash \alpha_i \in \theta$.

Since β_i is countable, and $\mathbb{P}$ is ω_1-closed, every subset of β_i in $M[\mathcal{G}]$ is in fact in M. Hence, there is $p_{i+1} < t_i$ and a subset B_i of β_i (in M), such that $p_{i+1} \Vdash \theta \cap \beta_i = \check{B}_i$.

Let $\alpha = \sup(\alpha_i) = \sup(\beta_i)$, $p_\omega = \cup p_i$. Since σ is closed, $p_\omega \Vdash \check{\alpha} \in \sigma$. Let $B = \cup_i B_i$. Then $p_\omega \Vdash \theta \cap \alpha = \check{B}$. Since $|p_\omega| = \alpha$, extending p_ω by B provides the required q. □

Corollary 3.6.13. *The Diamond principle is consistent with $\mathcal{ZFC}$.*

We note that in this model we will have $\mathcal{CH}$, even if M does not satisfy it. Since the extension preserves the power set, it must be that ω_1 becomes countable in this case.

By Theorem 1.4.17, we now know that the existence of a Suslin tree (i.e., the negation of the Suslin Hypothesis) is consistent with $\mathcal{CH}$. We show that it does not imply $\mathcal{CH}$.

Proposition 3.6.14. *Let T be an ever-branching Suslin tree in M, and let $\mathbb{P}$ be $\mathbb{F}n(A, 2)$ for some A. Then T is an ever-branching Suslin tree in $M[\mathcal{G}]$ for any generic $\mathcal{G}$.*

Proof. Assume not. Since $\mathbb{P}$ preserves ω_1, it is still the case that T is an ever-branching ω_1-tree in $M[\mathcal{G}]$. Hence, there must be an uncountable anti-chain, i.e., an injective function $f : \omega_1 \to T$ in $M[\mathcal{G}]$, such that $f(\alpha) \perp f(\beta)$ for $\alpha \neq \beta$. Let τ be a name for f, $p \in \mathbb{P}$ an element that forces this.

For each α, let $p_\alpha \in \mathcal{G}$ be an element that forces a particular value for $\tau(\alpha)$. By the Δ-system lemma (Lemma 2.3.7), applied in M, there is an uncountable subset I of ω_1 and a finite subset D such that p_α and p_β have the same restriction to their common domain D for all $\alpha \neq \beta \in I$. In particular, they are compatible. It now follows that τ produces an anti-chain of T of size ω_1 in M. This is a contradiction. □

Corollary 3.6.15. *If M has a Suslin tree, and for some cardinal κ in M, $\kappa^\omega = \kappa$, then M has a ctm extension that has a Suslin tree and where $2^\omega = \kappa$.*

Proof. By Proposition 3.6.14, Proposition 1.2.10 and Corollary 3.5.12, using the order $\mathbb{F}n(\kappa \times \omega, 2)$. □

End of lecture 25, Oct 26

4. Iterated forcing

In this section we discuss how to iterate the forcing construction. For example, suppose we wish to force two cardinal equalities, $2^\omega = \omega_5$ and $2^{\omega_1} = \omega_7$. To force the first one, we use $\mathbb{P} = \mathbb{F}n(\omega \times \omega_5, 2)$ to obtain a ctm N, which works by Corollary 3.5.12. For the second one, we may try two things: Using $\mathbb{Q}' = \mathbb{F}n(\omega_1 \times \omega_7, 2)$ in N will ensure $\omega_7 \leq 2^{\omega_1}$, but may add new subsets to ω, destroying the first equality.

We may also try $\mathbb{Q} = \mathbb{F}n(\omega_1 \times \omega_7, 2, \omega_1)$, which will preserve the power set of ω. From 3.6.8, we know that $\mathbb{Q}$ preserves cardinalities up to ω_1 and strictly above

$2^{<\omega_1}$. However, the last cardinal is at least ω_5 in N, so we know nothing about the preservation of ω_5. In fact, it will collapse ω_5 to ω_1: the surjective function $\omega_7 \to 2^{<\omega_1} = 2^\omega$ of the second point in 3.6.8 can be restricted to a surjective function on ω_1 (See Williams [8, Prop. 5.1] for details).

The correct approach turns out to be: use $\mathbb{Q}$ first. Starting from a model of $\mathcal{GCH}$ (where $\omega_7^{\omega_1} = \omega_7$), this forces $2^{\omega_1} = \omega_7$ by the third point in 3.6.8. On the other hand, by ω_1-closedness, we still have $2^\omega = \omega_1$, and therefore $\omega_5^\omega = \omega_5$.

We now use $\mathbb{P}$ on the resulting model. By the last equality, we may conclude that $2^\omega = \omega_5$ in a generic extension. It is also clear that we still have $2^{\omega_1} \geq \omega_7$ (note that cardinalities are preserved). Counting nice names again, one shows that $2^{\omega_1} \leq \omega_7$ as well.

Note that all of this trouble could be saved if we knew that the two extensions commute with each other. We will study ways of combining the two extensions into one extension, and then question of commuting becomes a question about different orders in one model. We begin with some generalities about comparing forcing with different orders.

4.1. **Comparing orders.** Let $\mathbb{P}$ and $\mathbb{Q}$ be two posets in M, and let $f : \mathbb{P} \to \mathbb{Q}$ be a poset map between them. We would like to compare forcing in $\mathbb{P}$ and in $\mathbb{Q}$. More precisely, given a generic filter $\mathcal{G}$ in $\mathbb{Q}$, we may hope to produce a generic filter in $\mathbb{P}$. Note that we have already considered this kind of questions in the proof of Theorem 2.3.11.

Theorem 4.1.1. *Let $f : \mathbb{P} \to \mathbb{Q}$ be a map of posets (in M).*

(1) *Assume that for any element $q \in \mathbb{Q}$, there is an element $p \in \mathbb{P}$ such that q is compatible with $f(p')$ for any $p' \leq p$ (p is called a* reduction *of q). Then for any dense D in $\mathbb{P}$, the set*
$$D' = \{q \in \mathbb{Q} | q \leq f(p) \quad \textit{for some } p \in D\} \tag{5}$$
is dense in $\mathbb{Q}$.

(2) *Assume in addition that whenever $x \perp y \in \mathbb{P}$, $f(x) \perp f(y)$. Then for any generic filter $\mathcal{G}$ in $\mathbb{Q}$, $\mathcal{F} = f^{-1}(\mathcal{G})$ is a generic filter in $\mathbb{P}$.*

We note that the sets D' are again in M.

Proof. (1) Given an element $q \in \mathbb{Q}$, let p be as in the assumption. Since D is dense, there is $p' \in D$ below p. By assumption, $f(p')$ is compatible with q, so there is some q' below both. Then $q' \in D'$.

(2) We claim first that $\mathcal{F}$ intersects any dense set in M. Otherwise, if D is such a set, we have $\mathcal{G} \cap f(D) = 0$, so by Lemma 3.4.3, there is an element $p \in \mathcal{G}$ incompatible with all elements of $f(D)$. This contradicts the fact that $\mathcal{G}$ is generic and D' is dense.

The rest of the proof is as in Theorem 2.3.11: given two elements $x, y \in \mathcal{F}$, the set $D_{xy} = \{z \in \mathbb{P} | z \leq x, y \vee z \perp x \vee z \perp y\}$ is dense (and in M), so intersects $\mathcal{F}$ at some element z. If, for instance, $z \perp x$, then $f(z) \perp f(x) \in \mathcal{G}$, a contradiction. Hence $z \leq x, y$, so x and y are compatible in $\mathcal{F}$.

□

Definition 4.1.2. A map of posets $f : \mathbb{P} \to \mathbb{Q}$ is called a *complete embedding* if it satisfies the two assumptions of Theorem 4.1.1.

Exercise 4.1.3. *The composition of two complete embeddings is a complete embedding.*

Example 4.1.4. *Let A, B be sets and κ a cardinal.*

(1) *If A' is a subset of A, then the inclusion $\mathbb{F}n(A', B, \kappa) \subseteq \mathbb{F}n(A, B, \kappa)$ is complete. The restriction $\mathbb{F}n(A, B, \kappa) \to \mathbb{F}n(A', B, \kappa)$ is not complete, since incompatible elements could become compatible.*
(2) *If $\kappa' > \kappa$ is another cardinal, $\mathbb{F}n(A, B, \kappa)$ is not a complete subset of $\mathbb{F}n(A, B, \kappa')$ (unless they are equal): a function in the latter but not in the former does not have a reduction.*
(3) *The first condition (reductions) is satisfied whenever $f(\mathbb{P})$ is dense in $\mathbb{Q}$. A complete embedding satisfying this condition is called a* dense embedding.

End of lecture 26, Oct 28

Dense embeddings give a more complete correspondence between generic filters. Let $f : \mathbb{P} \to \mathbb{Q}$ be an embedding. Define, for every filter $\mathcal{F}$ in $\mathbb{P}$, $f_*(\mathcal{F})$ to be the upward closure of $f(\mathcal{F})$.

Proposition 4.1.5. *Let $f : \mathbb{P} \to \mathbb{Q}$ be a dense embedding. The map $\mathcal{G} \mapsto f^{-1}(\mathcal{G})$ determines a one-to-one correspondence between generic filters in $\mathbb{P}$ and in $\mathbb{Q}$, with inverse $\mathcal{F} \mapsto f_*(\mathcal{F})$.*

Proof. We show that if $\mathcal{F}$ is generic, then so is $f_*(\mathcal{F})$. Given a dense subset D in $\mathbb{Q}$, let D' be the inverse image of its downwards closure. We note that D' is downward closed. We claim that D' is dense: given $p \in \mathbb{P}$, there is $d \in D$ below $f(p)$. Since f is dense, $f(p') \leq d$ for some $p' \in \mathbb{P}$. Then $p' \in D'$, and is compatible with p. Since D' is downward closed, we are done. It now follows that $\mathcal{F}$ intersects D', say at p. Then $f(p)$ is in $f_*(\mathcal{F})$ and below an element of D. Since $f_*(\mathcal{F})$ is a filter, it intersects D.

Given generic filters $\mathcal{F} \subseteq \mathbb{P}$ and $\mathcal{G} \subseteq \mathbb{Q}$, we now get inclusions of generic filters $\mathcal{F} \subseteq f^{-1}(f_*(\mathcal{F}))$ and $f_*(f^{-1}(\mathcal{G})) \subseteq \mathcal{G}$, which must be equalities by the following lemma. □

Lemma 4.1.6. *If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{P}$ are filters, and $\mathcal{F}$ is generic, then $\mathcal{F} = \mathcal{G}$.*

Proof. Otherwise, there is an element $p \in \mathcal{G} \setminus \mathcal{F}$. Apply Lemma 3.4.3 with $E = \{p\}$ and $\mathcal{F}$. □

Given a complete embedding, it makes sense to compare the corresponding ctms:

Proposition 4.1.7. *If $f : \mathbb{P} \to \mathbb{Q}$ is a complete embedding in M, and $\mathcal{G} \subseteq \mathbb{Q}$, then $M[\mathcal{G}]$ is a ctm extension of $M[f^{-1}(\mathcal{G})]$. If f is a dense embedding, they are the same.*

Proof. The first statement is obvious, since $f^{-1}(\mathcal{G})$ is in $M[\mathcal{G}]$. The second follows using Proposition 4.1.5. □

4.2. **Products.** We now deal with the simplified situation where both posets we wish to use for the extensions are already in M.

Definition 4.2.1. The *product* of two orders $\mathbb{P}$ and $\mathbb{Q}$ is the order on the set $\mathbb{P} \times \mathbb{Q}$ defined by $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 \leq p_2$ and $q_1 \leq q_2$. The maximal element is chosen to be $(\mathbf{1}_{\mathbb{P}}, \mathbf{1}_{\mathbb{Q}})$. It is again denoted by $\mathbb{P} \times \mathbb{Q}$.

Proposition 4.2.2. *The map $i_{\mathbb{P}} : \mathbb{P} \to \mathbb{P} \times \mathbb{Q}$, $i_{\mathbb{P}}(p) = (p, \mathbf{1})$ is a complete embedding. The product of a filter in $\mathbb{P}$ and a filter in $\mathbb{Q}$ is a filter in $\mathbb{P} \times \mathbb{Q}$, and any filter in $\mathbb{P} \times \mathbb{Q}$ is the product of its pullbacks along $i_{\mathbb{P}}$ and $i_{\mathbb{Q}}$.*

Proof. Exercise □

It follows that any generic filter in $\mathbb{P} \times \mathbb{Q}$ is a product of generic filters. The converse is not true, generally: for instance, if $\mathbb{Q} = \mathbb{P}$, and any element of p has two incompatible elements below it, then $D = \{(p, q)|p \perp q\}$ is dense, and intersect no filter of the form $\mathcal{G} \times \mathcal{G}$. The following theorem describes the situation.

End of lecture 27, Oct 31

Theorem 4.2.3. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets, containing filters $\mathcal{F}$ and $\mathcal{G}$, respectively. The following are equivalent:*

(1) *$\mathcal{F} \times \mathcal{G}$ is generic.*
(2) *$\mathcal{F}$ is generic, and $\mathcal{G}$ is generic over $M[\mathcal{F}]$.*

In this case, $M[\mathcal{F} \times \mathcal{G}] = M[\mathcal{F}][\mathcal{G}]$.

Of course, $\mathcal{F} \times \mathcal{G}$ is generic iff $\mathcal{G} \times \mathcal{F}$ is generic, so we may apply the Theorem in the other order.

Proof. Assume that $\mathcal{F} \times \mathcal{G}$ is generic. Then $\mathcal{F}$ is generic by completeness. Let σ be the name for a dense set of $\mathbb{Q}$ in $M[\mathcal{F}]$, $p_0 \in \mathcal{F}$ an element that forces it. Let

$$D' = \{(p, q) \in \mathbb{P} \times \mathbb{Q}|p \le p_0, p \Vdash q \in \sigma\} \tag{6}$$

We claim that D' is dense below $(p_0, 1)$: for any (p, q) with $p \le p_0$, let $q' \in \sigma_{\mathcal{F}}$ be below q. Then there is some $p' \le p$ that forces $q' \in \sigma$. Hence $(p', q') \in D'$.

Now, since $\mathcal{F} \times \mathcal{G}$ is generic, it intersects D' (note that D' is in M). Let (p, q) be in the intersection. Then, since $p \in \mathcal{F}$, we get that $q \in \sigma_{\mathcal{F}}$ in $M[\mathcal{F}]$. Since also $q \in \mathcal{G}$, we are done.

In the other direction, let D be dense in $\mathbb{P} \times \mathbb{Q}$ (over M), and let D' be the projection of $D \cap \mathcal{F} \times \mathbb{Q}$ to $\mathbb{Q}$. Note that D' is in $M[\mathcal{F}]$. It is also dense: given $q \in \mathbb{Q}$, let $E = \{p \in \mathbb{P}|(p, q') \in D \text{ for some } q' \le q\}$. Then E is dense in $\mathbb{P}$ since D is dense, so there is some $p \in \mathcal{F} \cap E$. For some $q' \le q$ we thus have $(p, q') \in D$, so $q' \in D'$.

We now have that $\mathcal{G}$ intersects D'. Let q be in the intersection. Then it comes from some $(p, q) \in D \cap \mathcal{F} \times \mathbb{Q}$. Hence $(p, q) \in D \cap \mathcal{F} \times \mathcal{G}$.

The last statement follows from the fact that both sides contains $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$. □

One application of this Theorem is to reduce problems about forcing with $\mathbb{F}n(A, 2)$ to forcing with $\mathbb{F}n(B, 2)$, where $B \subseteq A$ has smaller cardinality. This is achieved using the following result.

Lemma 4.2.4. *Let I, S be sets in M, let $\mathbb{P} = \mathbb{F}n(I, 2)$, let $\mathcal{G}$ be a generic filter, and let A be a subset of S in $N = M[\mathcal{G}]$. Then there is a subset I_0 of I of cardinality at most $|S|$ (all computed in M), such that $N = M[\mathcal{G}_0][\mathcal{G}_1]$, with A in $M[\mathcal{G}_0]$, and $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{F}n(I_0, 2)$.*

Proof. Let σ be a nice name for A. Hence σ has domain S, and for each $s \in S$, the fibre σ_s is an anti-chain, hence countable by the ccc. Hence the union I_0 of the domains of all elements appearing in σ is of cardinality at most $|S|$.

We get the result using the Theorem above, noting that $\mathbb{F}n(I, 2)$ is isomorphic to $\mathbb{F}n(I_0, 2) \times \mathbb{F}n(I \setminus I_0, 2)$. □

As an application, we prove the consistency of a weak version of $\mathcal{MA}$.

Definition 4.2.5. The axiom $\mathcal{MAC}(\kappa)$ is Martin's axiom $\mathcal{MA}(\kappa)$ restricted to countable posets (likewise for $\mathcal{MAC}$).

The advantage of using countable posets is that they all look the same in terms of forcing (unless they are trivial). More precisely:

Proposition 4.2.6. *Let $\mathbb{P}$ and $\mathbb{Q}$ be two countable atomless posets in M. If $\mathcal{G}$ is a generic filter in $\mathbb{Q}$, then $\mathbb{P}$ has a generic filter in $M[\mathcal{G}]$.*

It follows that each determines the same forcing extensions as the other.

Proof. It is enough, by Proposition 4.1.5, to find an order that densely embeds in both. We assume $\mathbb{P}$ to be the subset of $\mathbb{F}n(\omega,\omega)$ consisting of functions whose domain is an initial segment of ω, and constructed a dense embedding $i : \mathbb{P} \to \mathbb{Q}$, by induction on the domain. Let q_i be an enumeration of $\mathbb{Q}$.

We must have $i(0) = \mathbf{1}_{\mathbb{Q}}$. Let $p \in \mathbb{P}$ be an element with domain n. Since $\mathbb{Q}$ is atomless, there is an infinite maximal anti-chain t_i of elements below $i(p)$, and if q_n is compatible with $i(p)$, we may choose $t_0 \le q_n$. We set $i(p \cup (n,m)) = t_m$. To show that it is a dense embedding, it is enough to show that there is p as above such that $i(p)$ is compatible with q_n. This is clear by maximality of the anti-chain. □

End of lecture 28, Nov 2

Corollary 4.2.7. *If $\mathbb{P}$ is countable and A is infinite, both in M, and $\mathcal{G}$ is $\mathbb{F}n(A,2)$-generic, then $\mathbb{P}$ has a generic filter in $M[\mathcal{G}]$.*

Proof. By Lemma 4.2.4, $M[\mathcal{G}] = M[\mathcal{G}_0][\mathcal{G}_1]$, where $\mathcal{G}_0 = \mathcal{G} \cap \mathbb{F}n(A_0,2)$ with $A_0 \subseteq A$ countable. By Proposition 4.2.6, $\mathbb{P}$ has a generic filter in $M[\mathcal{G}_0]$. □

Theorem 4.2.8. *Assume that $\kappa^\omega = \kappa$ in M, and let $\mathcal{G}$ be generic in $\mathbb{F}n(\kappa,2)$. Then $M[\mathcal{G}]$ satisfies $\mathcal{MAC}$.*

Recall that $\mathbb{F}n(\kappa,2)$ forces $2^\omega = \kappa$, so this shows that $\mathcal{MAC}$ is consistent with the negation of $\mathcal{CH}$. Also, in the proof, we need to verify $\mathcal{MAC}(\lambda)$ for $\lambda < \kappa$.

Proof. Fix $\lambda < \kappa$, a countable poset $\mathbb{P}$, and a family of dense subsets D_α, $\alpha < \lambda$, all in $M[\mathcal{G}]$. Note that M also thinks that $\mathbb{P}$ is countable. By Lemma 4.2.4, we may write $M[\mathcal{G}] = M[\mathcal{G}_0][\mathcal{G}_1]$, where $\mathcal{G}_0 \subseteq \mathbb{F}n(I_0,2)$, $|I_0| \le \lambda < \kappa$, and $\mathbb{P}$ and the D_α are in $M[\mathcal{G}_0]$. Since $\mathbb{P}$ is countable, we are done by Corollary 4.2.7. □

To what extent is $\mathcal{MAC}(\kappa)$ weaker than $\mathcal{MA}(\kappa)$? Recall that under $\mathcal{MA}(\kappa)$ we have

- Any family of functions $\omega \to \omega$ of cardinality κ is dominated (Proposition 2.2.2).
- Any mad family is of cardinality bigger than κ (Proposition 2.2.6)

For the first point, we may define

Definition 4.2.9. The *bounding number* $\mathfrak{b}$ is the least size of a family that is not dominated by any function. The *dominating number* is the least size of a family $\mathscr{F}$ such that each function is dominated by some element of $\mathscr{F}$.

Thus, $\mathcal{MA}(\kappa)$ implies that $\mathfrak{b} > \kappa$. If $\mathscr{F}$ is dominated by a function g, then g is a function that is not dominated by any element of $\mathscr{F}$. Hence $\mathfrak{b} \le \mathfrak{d}$. We have

Exercise 4.2.10. *$\mathcal{MAC}(\kappa)$ implies $\mathfrak{d} > \kappa$ (use $\mathbb{F}n(\omega,\omega)$).*

So the question is: Could we have $\mathfrak{d} > \mathfrak{b}$?

For the second point, we define

Definition 4.2.11. An *independent family* is a family $\mathscr{F}$ of subsets of ω such that each finite non-empty boolean combination of distinct elements of $\mathscr{F}$ is infinite.

Exercise 4.2.12. *$\mathcal{MAC}(\kappa)$ implies that any maximal independent family is of size more than κ (use $\mathbb{F}n(\omega, 2)$).*

Again, we can ask: Can we have a small mad family, but no small maximal independent family?

By the exercises, it is enough to show that in an extension using $\mathbb{F}n(\kappa, 2)$ there are a mad family, and a family of un-dominated functions of size ω_1. This will also show the consistency of $\mathcal{MAC}$ and the negation of $\mathcal{MA}$ and $\mathcal{CH}$.

Theorem 4.2.13. *Let M be a ctm satisfying $\mathcal{CH}$ and $\kappa^\omega = \kappa$ and let $\mathcal{G}$ be a generic filter in $\mathbb{F}n(\kappa, 2)$. Then in $M[\mathcal{G}]$ there is a family of functions $\omega \to \omega$ not dominated by any function, and a mad family, both of size ω_1.*

Proof. In both cases, we use the product theorem to reduce to a forcing extension with $\mathbb{F}n(\omega, 2)$. For the first problem, it is enough to find a family of functions in M of size ω_1, which is not dominated in $M[\mathcal{G}]$. Since in M $\omega_1 = 2^\omega$, we may as well take all functions from ω to itself (in M). If this family is dominated in $M[\mathcal{G}]$ by some function g, then by Lemma 4.2.4, $g \subseteq \omega \times \omega$ already occurs in $M[\mathcal{F}]$, where $\mathcal{F}$ is a generic filter in $\mathbb{F}n(I, 2)$ for some countable $I \subseteq \kappa$. Hence the result will follow from Lemma 4.2.14 below.

Similarly, to construct a mad family of size ω_1 in $M[\mathcal{G}]$, it is enough to construct any mad family in M (since M satisfies $\mathcal{CH}$), so that it remains maximal in $M[\mathcal{G}]$. A counter-example to maximality will be a subset of ω, so will occur already in $M[\mathcal{F}]$, with $\mathcal{F}$ as above. Hence the result follows from Lemma 4.2.15. □

Lemma 4.2.14. *Let M be a ctm, and let $\mathcal{G}$ be generic in $\mathbb{P} = \mathbb{F}n(\omega, 2)$. Then any $g : \omega \to \omega$ in $M[\mathcal{G}]$ does not dominate some $f : \omega \to \omega$ in M.*

Proof. We may assume (by fixing a bijection in M) that we are dealing with functions $\mathbb{P} \to \omega$ instead. Let σ be a name for g. Define $f(p) = n + 1$ if $p \Vdash \sigma(p) = n$, $f(p) = 0$ otherwise. Note that this is defined in M. Since any actual value of g is forced by some $p \in \mathcal{G}$, f is not dominated. □

Lemma 4.2.15. *Let M be a ctm satisfying $\mathcal{CH}$, and let $\mathcal{G}$ be generic in $\mathbb{F}n(\omega, 2)$. There is a mad family in M that is still maximal in $M[\mathcal{G}]$.*

Proof. There are $\omega^\omega = \omega_1$ nice names for subsets of ω (in M). Let (τ_α, p_α) for $\omega \leq \alpha < \omega_1$ be an enumeration of all pairs of a nice name and a condition. We define a mad family A_α by induction, so that p_α force τ_α to interesect some A_β at an infinite set ($\beta \leq \alpha$). We choose the A_n for $n < \omega$ as any partition of ω into infinite sets.

Assume A_β was defined for $\beta < \alpha$. If p_α does not force τ_α to be infinite, or the intersection of τ_α with all A_β to be finite, there is nothing to do, and we choose A_α in any way that keeps the family almost disjoint. Hence we assume that p_α forces τ_α to be infinite and all such intersections to be finite, and we would like to construct A_α so that p_α forces the intersection of A_α with τ_α to be infinite.

Let B_i, $i < \omega$ enumerate all A_β, $\beta < \alpha$, and let (n_i, q_i), $i < \omega$ enumerate $\omega \times \{q \leq p_\alpha\}$. By assumption, each q_i forces $|\tau_\alpha \setminus \bigcup_{j<i} B_j| = \omega$. Hence we may

find $m_i > n_i$ and $r_i \leq q_i$ such that $r_i \Vdash m_i \in \tau_\alpha$, but $m_i \notin B_j$ for $j < i$. We let $A_\alpha = \{m_i\}$. Clearly A_α is infinite, but intersects each B_i (i.e., each A_β) in a finite set. Also, p_α forces τ_α to intersect A_α in an infinite subset. □

End of lecture 29, Nov 7

4.2.16. *Easton forcing.* We are interested obtain a more complete picture of the possible cardinalities of power sets. Let M be a model of $\mathcal{ZFC}$, and let E be the "exponential function" $\kappa \mapsto 2^\kappa$. Then E satisfies:

1. E is weakly increasing
2. For all κ, $\mathrm{cf}(E(\kappa)) > \kappa$ (König's Theorem)

It turns out that, at least for regular cardinals, these are the only restrictions.

Definition 4.2.17. An *Easton index function* is a function E on a set of regular cardinals satisfying the conditions above.

Given a set D of cardinals and a cardinal λ, we denote by $D_{\prec\lambda} = \{\kappa \in D | \kappa \prec \lambda\}$, where $\prec$ is $<$, $\leq$, $>$, etc. To force E to be an actual exponential, we will use the following order.

Definition 4.2.18. Given an Easton function E with domain D, $\mathbb{P}(E)$ is the poset of partial functions p on D with $p(\kappa) \in \mathbb{F}n(E(\kappa), 2, \kappa)$ for all $\kappa \in D$, and such that for any regular λ (not only in D), the support of p restricted to $D_{<\lambda}$ is smaller than λ.

Example 4.2.19. *Let $D = \{\omega, \omega_1\}$, $E(\omega) = \omega_5$ and $E(\omega_1) = \omega_7$. Then E is an Easton index function, and $\mathbb{P}(E) = \mathbb{F}n(\omega_5, 2) \times \mathbb{F}n(\omega_7, 2, \omega_1) = \mathbb{P} \times \mathbb{Q}$ in the terminology of the beginning of this section.*

We fix an Easton index function E, with domain D. In general, $\mathbb{P}(E)$ encodes a plan for applying 3.6.8 for all $\mathbb{F}n(E(\kappa), 2, \kappa)$ with $\kappa \in D$ at once.

We also fix a cardinal λ. To study the effect of forcing with $\mathbb{P}(E)$ on λ, we present it as a product, as follows.

Definition 4.2.20. E^+ and E^- are the restrictions of E to $D_{>\lambda}$ and to $D_{\leq\lambda}$, respectively.

Lemma 4.2.21. *With E and λ as above*

1. $\mathbb{P}(E) = \mathbb{P}(E^+) \times \mathbb{P}(E^-)$
2. $\mathbb{P}(E^+)$ *is λ^+-closed*

Proof. Exercise □

The Lemma will ensure preservation of "small" cofinalities. To obtain preservation of "large" cofinalities, we need, as before, an (anti-) chain condition.

Proposition 4.2.22. *Assume that λ is regular. Then $\mathbb{P}(E^-)$ is $(2^{<\lambda})^+$-c.c.*

Proof. An element $p \in \mathbb{P}(E^-)$ determines a partial function $f(\kappa, \theta) = p(\kappa)(\theta)$ for $\kappa \in D_{\leq\lambda}$ and $\theta \in E(\kappa)$. Since the domain of definition of $p(\kappa)$ has cardinality smaller than κ, we have that the domain of f is of size smaller than $\lambda \times \lambda = \lambda$. In other words, $\mathbb{P}(E^-)$ can be viewed as a sub-order of $\mathbb{Q} = \mathbb{F}n(D_{\leq\lambda} \times \lambda, 2, \lambda)$, which has the $(2^{<\lambda})^+$-c.c. by Corollary 3.6.10. Since incompatible elements in $\mathbb{P}(E^-)$ remain incompatible in $\mathbb{Q}$, we get the result. □

End of lecture 30, Nov 9

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Corollary 4.2.23. *Assume M satisfies $\mathcal{GCH}$. Then $\mathbb{P}(E)$ preserves cofinalities (and hence cardinals).*

Proof. Otherwise, there is some uncountable regular θ in M that becomes singular in $M[\mathcal{G}]$, i.e., we have, in $M[\mathcal{G}]$ an increasing cofinal function $f : \lambda \to \theta$ (Lemma 3.5.5), where $\lambda < \theta$ is regular in $M[\mathcal{G}]$, and hence in M. We use λ to write $\mathbb{P} = \mathbb{P}(E^-) \times \mathbb{P}(E^+) = \mathbb{P}^- \times \mathbb{P}^+$, so $\mathcal{G} = \mathcal{G}^- \times \mathcal{G}^+$, and $M[\mathcal{G}] = M[\mathcal{G}^+][\mathcal{G}^-]$.

Let $N = M[\mathcal{G}^+]$. Since $\mathbb{P}^+$ is λ^+-closed, any function with domain λ comes from M. Hence f is not in N. Furthermore, in N $\mathcal{GCH}$ holds up to λ. In particular, in N, $2^{<\lambda} = \lambda$. Hence, by Proposition 4.2.22, $\mathbb{P}^-$ is λ^+-closed in N. It follows, as in Theorem 3.5.4, that it preserves cofinalities above λ. In particular, θ should be regular in $N[\mathcal{G}^-] = M[\mathcal{G}]$. □

Corollary 4.2.24. *If M satisfies $\mathcal{GCH}$, and $\mathcal{G}$ is generic in $\mathbb{P}(E)$, then $2^\lambda = E(\lambda)$ for all $\lambda \in D$.*

Proof. $2^\lambda \geq E(\lambda)$ is exactly as before. For the other inequality, split $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$, $\mathcal{G} = \mathcal{G}^- \times \mathcal{G}^+$, $M[\mathcal{G}] = M[\mathcal{G}^+][\mathcal{G}^-]$ as in the previous proof. By $\mathcal{GCH}$ in M, we have (in M), $|\mathbb{F}n(E(\kappa), 2, \kappa)| \leq E(\lambda)$ for $\kappa \leq \lambda$, so $|\mathbb{P}^-| = E(\lambda)$. Since $\mathbb{P}^+$ is λ^+-closed, we have $\mathcal{GCH}$ up to λ in $M[\mathcal{G}^+]$. In particular, $\mathbb{P}^-$ is λ^+-cc in $M[\mathcal{G}^+]$, so we may count nice names. □

Remark 4.2.25. It is also possible to compute the cardinality of power sets 2^κ for $\kappa \notin D$: Given an Easton index function E on D, there is an extension E' of E to all cardinals, such that E' is pointwise minimal. Then in $M[\mathcal{G}]$ as above, $2^\kappa = E'(\kappa)$ for all κ. Our proof essentially shows it in the case that κ is regular. The general case involves some computations, see Kunen [4, § VIII.4.7].

4.3. **General iterated forcing.** We now arrive at the general case: starting with a poset $\mathbb{P}$ in M, and a generic filter $\mathcal{F}$, we have the ctm $N = M[\mathcal{F}]$. Assume now that we have a poset $\mathbb{Q}$ in N, and a generic filter $\mathcal{G}$ in it, so we obtain a third model $N[\mathcal{G}]$. We would like to come up with a poset $\mathbb{P}_{\mathbb{Q}}$ in M, and a generic filter $\mathcal{F}_{\mathcal{G}}$ in $\mathbb{P}_{\mathbb{Q}}$, such that $M[\mathcal{F}_{\mathcal{G}}] = M[\mathcal{F}][\mathcal{G}]$.

In fact, we will construct something more uniform. The poset $\mathbb{Q}$ is not (in general) in M, but it has a name π in M. Given any such name, we construct a poset $\mathbb{P} * \pi$ and a map $i : \mathbb{P} \to \mathbb{P} * \pi$, both in M, satisfying the following

4.3.1. *Conditions.*

(1) $i : \mathbb{P} \to \mathbb{P} * \pi$ is a complete embedding
(2) Given a generic filter $\mathcal{F}$ of $\mathbb{P}$, any filter $\mathcal{G}$ of $\pi_{\mathcal{F}}$ determines a filter $\mathcal{F} * \mathcal{G}$ of $\mathbb{P} * \pi$
(3) Given a generic filter $\mathcal{K}$ of $\mathbb{P} * \pi$, let $\mathcal{F} = i^{-1}(\mathcal{K})$. Then $\mathcal{K}$ determines a generic $\mathcal{G}$ of $\pi_{\mathcal{F}}$, such that $\mathcal{K} = \mathcal{F} * \mathcal{G}$, and $M[\mathcal{K}] = M[\mathcal{F}][\mathcal{G}]$.

We note that it does not make sense to ask for symmetry here. We also note that $\mathbb{P} * \pi$ is an actual poset, not the name of a poset. This will allow us to iterate the construction.

We now construct what we promised. By a statement like "π is a name for a poset", or "σ is a name for a subset of π", we mean that this hold definably. For example, in the first case, there are names ρ and σ (fixed, even if they are not mentioned), such that for any generic $\mathcal{G}$, $\rho_{\mathcal{G}} \subseteq \pi_{\mathcal{G}} \times \pi_{\mathcal{G}}$, and is a partial order there, and $\sigma_{\mathcal{G}} \in \pi_{\mathcal{G}}$, and is a maximal element. Equivalently, **1** forces the corresponding conditions. Given, e.g., a partial order in some $M[\mathcal{G}]$, it follows from the basic construction (essentially, existence of Skolem functions), that we may find a name that is definably a poset in the above sense. Furthermore, when σ is the name of a subset of π (or the name of an order on π, etc.), we may (and will) assume that $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\pi)$ (for example by nice names, Lemma 3.5.10).

Definition 4.3.2. Let $\mathbb{P}$ be a poset in M, and let π be the name for a poset. We set

$$\mathbb{P} * \pi = \{(p, \tau) | p \in \mathbb{P}, \tau \in \mathrm{dom}(\pi), p \Vdash \tau \in \pi\} \tag{7}$$

This set is ordered by: $(p, \tau) \leq (q, \sigma)$ if $p \leq q$ and $p \Vdash \tau \leq \sigma$. The maximal element is $(\mathbf{1}_{\mathbb{P}}, \mathbf{1}_\pi)$ (where $\mathbf{1}_\pi$ is a name for the maximal element of π, whose domain is in $\mathrm{dom}(\pi)$).

The map $i : \mathbb{P} \to \mathbb{P} * \pi$ is defined by $i(p) = (p, \mathbf{1}_\pi)$.

Exercise 4.3.3. *When $\mathbb{Q}$ is another order in M, and $\pi = \check{\mathbb{Q}}$, we recover the product (up to isomorphism).*

Example 4.3.4. *Assume that $\mathcal{G}$ is generic in some $\mathbb{P}$, and let $\mathbb{Q} = \mathbb{F}n(\kappa, 2, \omega_1)$ in $M[\mathcal{G}]$. Any element of $\mathbb{Q}$ has a name τ in M, and we may even ask that $\mathbf{1}_{\mathbb{P}}$ forces it and that $\mathrm{dom}(\tau) \subseteq \mathrm{dom}(\kappa \times 2)$. If S is the set of all such names, we therefore have that $\pi = S \times \{\mathbf{1}_{\mathbb{P}}\}$ is a name for $\mathbb{Q}$. Hence $\mathbb{P} * \pi$ is the set of pairs (p, τ) with τ as above. We note that in this example, the order is not strict.*

Exercise 4.3.5. *i is a complete embedding*

Definition 4.3.6. Let $\mathbb{P}$ be a poset, and let π be a name for a poset

(1) If $\mathcal{G}$ is generic in $\mathbb{P}$, and $\mathcal{H}$ is a filter in $\pi_{\mathcal{G}}$, then $\mathcal{G} * \mathcal{H} \subseteq \mathbb{P} * \pi$ is the set $\{(p, \tau) | p \in \mathcal{G}, \tau_{\mathcal{G}} \in \mathcal{H}\}$.
(2) If $\mathcal{K}$ is a generic filter in $\mathbb{P} * \pi$, let $\mathcal{G} = i^{-1}(\mathcal{K})$ (so $\mathcal{G}$ is generic). Then $\mathcal{K}_{\mathcal{G}} = \{\tau_{\mathcal{G}} | \exists p((p, \tau) \in \mathcal{K})\}$

Exercise 4.3.7. *Show that $\mathcal{G} * \mathcal{H}$ is a filter in $\mathbb{P} * \pi$ and $\mathcal{K}_{\mathcal{G}}$ is a filter in $\pi_{\mathcal{G}}$ (in the assumptions of the above definition).*

Theorem 4.3.8. *If $\mathcal{K} \subseteq \mathbb{P} * \pi$ is generic, $\mathcal{G} = i^{-1}(\mathcal{K})$ and $\mathcal{H} = \mathcal{K}_{\mathcal{G}}$, then $\mathcal{G}$ and $\mathcal{H}$ are generic, $\mathcal{K} = \mathcal{G} * \mathcal{H}$, and $M[\mathcal{K}] = M[\mathcal{G}][\mathcal{H}]$.*

Proof. $\mathcal{G}$ is generic by Exercise 4.3.5, $\mathcal{K} = \mathcal{G} * \mathcal{H}$ is an exercise, and $M[\mathcal{K}] = M[\mathcal{G}][\mathcal{H}]$ is obvious (given the other stuff). The proof that $\mathcal{H}$ is generic is exactly as in the proof of 4.2.3. □

End of lecture 31, Nov 11

For the applications, we need to know what happens with the cc conditions when taking products. Recall that if $\mathbb{P}$ has κ-cc, then it preserves cofinalities starting from κ. Hence, if σ is the name of a subset of κ that has smaller cardinality (i.e., $\mathbf{1}_{\mathbb{P}}$ forces it), then, for any generic filter $\mathcal{G}$ there is some ordinal $\beta < \kappa$ such that $\sigma_{\mathcal{G}} \subset \beta$. The following lemma says that β can be picked independently of $\mathcal{G}$.

Lemma 4.3.9. *Assume that $\mathbb{P}$ is κ-cc for regular κ, and that σ is a name with $\mathbf{1} \Vdash (\sigma \subseteq \kappa \wedge |\sigma| < \kappa)$. Then $\mathbf{1} \Vdash \sigma \subseteq \beta$ for some $\beta < \kappa$.*

Proof. Let E be the set of possible values of $\alpha = \sup(\sigma_{\mathcal{G}})$. Each such value is forced forced by some $p_\alpha \in \mathbb{P}$, and since they are an anti-chain, $|E| < \kappa$. Take $\beta = \sup E$. □

Proposition 4.3.10. *Let κ be regular, assume that $\mathbb{P}$ is κ-cc, and that π is the name of a κ-cc poset. Then $\mathbb{P} * \pi$ is κ-cc.*

Proof. Let $\{(p_\alpha, \tau_\alpha)\}$ be an anti-chain indexed by elements κ, and let σ be the name $\{(\alpha, p_\alpha)\}$. Hence $\mathbf{1} \Vdash \sigma \subseteq \kappa$.

Let $\mathcal{G}$ be generic in $\mathbb{P}$, $S = \sigma_\mathcal{G}$. Then $C = \{(\tau_\alpha)_\mathcal{G} | \alpha \in S\}$ is an anti-chain in $\pi_\mathcal{G}$: if $\theta_\mathcal{G}$ is below $(\tau_\alpha)_\mathcal{G}$ and $(\tau_\beta)_\mathcal{G}$, this is forced by some $p \in \mathcal{G}$, which we may assume below p_α, p_β. Then (p, θ) is below (p_α, τ_α), (p_β, τ_β).

It follows that the $|C| < \kappa$, so $\mathbf{1} \Vdash |\sigma| < \kappa$. By the previous Lemma, σ is uniformly contained in some $\beta < \kappa$, so $|\sigma| < \kappa$. □

Note that this does *not* imply that the product of (say) ccc orders is ccc, since the second order may cease being ccc after a generic extension for the first.

We pass to general iterations. We will define, for an ordinal α, the notion of an α-stage forcing construction. The definition is by induction on α, where for successor it is essentially the product just defined. For limit ordinals, we are essentially taking a limit, but as with Easton forcing, we would like to have a control of the support. Hence, we need also to define the support of an element, which will be a subset of α. The collection of admissible supports will be given by an ideal in $\mathcal{P}\alpha$.

Definition 4.3.11. Let $\mathcal{I}$ be an ideal in $\mathcal{P}\alpha$ (not necessarily proper), containing all finite sets. An *α-stage iterated forcing construction with supports in $\mathcal{I}$* is a pair $(\mathbb{P}, \pi)$, where $\mathbb{P}$ is a function on $\alpha + 1$, and π is a function on α, satisfying

(1) For each β, $\mathbb{P}_\beta$ is a poset in M, and π_β is a $\mathbb{P}_\beta$-name for a poset.
(2) $\mathbb{P}_0 = 1$.
(3) For $\alpha = \beta + 1$, the restriction to $\beta + 1$ and β, respectively, is a β-stage construction with support in $\mathcal{I}$, and $\mathbb{P}_\alpha = \mathbb{P}_\beta * \pi_\beta$.
(4) When α is limit, $\mathbb{P}_\alpha$ consists of elements of the inverse limit $\varprojlim_{\beta<\alpha} \mathbb{P}_\beta$ (with respect to the projection maps), whose support is in $\mathcal{I}$ (the support is defined below).

Hence, for any $\beta < \gamma < \alpha + 1$, there is a projection map $\Pi_{\gamma,\beta} : \mathbb{P}_\gamma \to \mathbb{P}_\beta$. For $x \in \mathbb{P}_\gamma$, we denote $\Pi_{\gamma,\beta}(x)$ by $x\restriction_\beta$.

The *support* $|x|$ of an element of $x \in \mathbb{P}_\alpha$ is a subset of α defined as follows

(1) When $\alpha = \beta + 1$, $|x| = |x\restriction_\beta|$ if $x(\beta) = \mathbf{1}_{\mathbb{P}_\beta}$, $|x\restriction_\beta| \cup \{\beta\}$ otherwise.
(2) When α is limit, $|x| = \sup\{|x\restriction_\beta| | \beta < \alpha\}$

Remark 4.3.12. (1) One says "finite support", "countable support" or "full limits" when $\mathcal{I}$ is the ideal of finite sets, countable sets, or all subsets, respectively.
(2) For $\alpha = 1$, we just have a poset. For $\alpha = 2$, this is the product discussed above
(3) An element of $\mathbb{P}_\alpha$ can be viewed as a function on α in the obvious way, and then support coincides with what we defined before.
(4) For each $\beta \le \alpha$ there is a map $i_{\beta,\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$, defined inductively by: $i_{\beta,\beta}$ is the identity, $i_{\beta,\alpha+1} = i \circ i_{\beta,\alpha}$, where i is the map given by the 2-stage step (for limits, use universal property). Since the composition of complete embeddings is complete, each of these is complete (exercise for limits).

End of lecture 32, Nov 14

Example 4.3.13. *Usual infinite products are a special case (where we have* $\pi_\alpha = \check{\mathbb{Q}}_\alpha$ *for some posets* $\mathbb{Q}_\alpha$ *in* M*). For instance, forcing with an Easton index function* E *on* $D = \{\kappa_\beta | \beta < \alpha\}$ *can be viewed as an* α*-stage forcing with support in* $\mathcal{I}$*, where* $\pi_\beta = \mathbb{F}n(E(\check{\kappa_\beta}), 2, \kappa_\beta)$*, and* $\mathcal{I}$ *is the ideal of subsets* $S \subseteq \alpha$ *such that* $|\{\kappa_\beta < \lambda | \beta \in S\}| < \lambda$ *for all regular* λ*.*

If $\mathbb{P}$ is an α-forcing construction, we write $\Vdash_\beta$ for the forcing relation with respect to $\mathbb{P}_\beta$ ($\beta \leq \alpha$). We note that any $\mathbb{P}_\beta$-name can also be viewed as a $\mathbb{P}_\alpha$-name, by applying $i_{\beta,\alpha}$.

We now generalise some previous results that we had for usual products and for 2-step forcing to the general case. The following is a straightforward application of Theorem 4.3.8.

Proposition 4.3.14. *Let* α*,* $\mathcal{I}$ *and* $\mathbb{P}_\mathcal{I}$ *be as in the definition, and let* $\mathcal{G} \subseteq \mathbb{P}_\mathcal{I}$ *be a generic filter. If* $\beta < \alpha$*, then* $\mathcal{G}_\beta := {i_{\beta,\alpha}}^{-1}(\mathcal{G})$ *is generic in* $\mathbb{P}_\beta$*, for* $\gamma \leq \beta$ $M[\mathcal{G}_\gamma] \subseteq M[\mathcal{G}_\beta]$*, and if* $\mathcal{H}_\beta = {\mathcal{G}_{\beta+1}}_{\mathcal{G}_\beta}$ *(in the sense of Definition 4.3.6), then* $\mathcal{H}_\beta$ *is generic in* ${\pi_\beta}_{\mathcal{G}_\beta}$*, and* $M[\mathcal{G}_{\beta+1}] = M[\mathcal{G}_\beta][\mathcal{H}_\beta]$*.*

Proof. Exercise □

We also have a general version of Lemma 4.2.4

Proposition 4.3.15. *Let* S *be set in* M*,* α*,* $\mathbb{P}_\mathcal{I}$ *be as above, and assume that each element of* $\mathcal{I}$ *is bounded in* α*. If* $\mathcal{G}$ *is generic in* $\mathbb{P}_\mathcal{I}$*, such that* $|S| < \mathrm{cf}(\alpha)$ *in* $M[\mathcal{G}]$ *and* A *is a subset of* S *in* $M[\mathcal{G}]$*, then* $A \in M[\mathcal{G}_\beta]$ *for some* $\beta < \alpha$*.*

Proof. Exercise (mimic 4.2.4: Note that $\mathbb{F}n(\alpha, 2)$ is an α stage forcing with finite supports, where each π_β is a linear order with 3 elements). □

Finally, we need a chain condition:

Proposition 4.3.16. *Let* α *be an ordinal,* $\mathbb{P}$ *an* α*-forcing with finite supports, and suppose that for each* $\beta < \alpha$*,* π_β *is the name for a* κ*-cc poset. Then* $\mathbb{P}$ *is* κ*-cc.*

Proof. Exercise (Use Proposition 4.3.10 for successor stages, the Δ-system lemma for limits). □

We are now ready to prove the independence of $\mathcal{MA}$. The proof is very similar to the proof of consistency of $\mathcal{MAC}$ (Theorem 4.2.8), but with more indices. What helped us in the countable case was the fact that there is essentially just one non-trivial countable poset, so one generic extension included a generic filter in all of them. This is not the case for general posets, but we will construct a sequence that includes all of them explicitly.

For the duration of the following lemma, we define a *relevant name* to mean a name for a ccc partial order on a cardinal smaller than κ.

Lemma 4.3.17. *Let* κ *be a regular cardinal, such that* $2^{<\kappa} = \kappa$*. There is a* κ*-stage forcing construction* $(\mathbb{P}, \pi)$ *with the following properties.*

(1) *Each* $\mathbb{P}_\alpha$ *is ccc, and each* π_α *is a relevant name*
(2) $|\mathbb{P}_\alpha| < \kappa$ *for* $\alpha < \kappa$*, and* $|\mathbb{P}_\kappa| \leq \kappa$*.*
(3) *If* τ *is a relevant* $\mathbb{P}_\alpha$*-name, then there is* $\beta \geq \alpha$ *such that* $\mathbf{1} \Vdash_\beta \tau = \pi_\beta$

Proof. We fix a surjective map $f : \kappa \to \kappa \times \kappa$, such that if $f(\beta) = (\alpha, \gamma)$ then $\beta \geq \alpha$. We define, by induction on α, π_α and relevant $\mathbb{P}_\alpha$-names σ^α_γ, $\alpha, \gamma \in \kappa$ that enumerate names for all possible partial orders on cardinals $< \kappa$.

Assume the data was defined for all $\beta < \alpha$, and let $f(\alpha) = (\beta, \gamma)$. Then σ^β_γ is a well-defined $\mathbb{P}_\beta$-name. Let $\sigma = i_{\beta,\alpha}(\sigma^\beta_\gamma)$. We may assume (using Skolem functions) that σ is again a relevant name with respect to $\mathbb{P}_\alpha$. This is π_α. Next, we let $(\sigma^\alpha_\gamma)_\gamma$ be an enumeration of all nice relevant $\mathbb{P}_\alpha$-names. By the induction hypothesis, $|\mathbb{P}_\alpha| < \kappa$, and $\mathbb{P}_\alpha$ is ccc, so there are at most $\kappa^{\omega \times \lambda} = \kappa$ nice names for subsets of $\lambda \times \lambda$. Hence $\gamma \in \kappa$.

This completes the construction (at stage α). We now verify the properties. π_α is a relevant name by construction. By Proposition 4.3.10 (and the induction hypothesis), $\mathbb{P}_{\alpha+1}$ is ccc (For the limit stages, we use Proposition 2.3.6). To compute the cardinality of $\mathbb{P}_{\alpha+1}$, assume that π_α is the name of an order on $\lambda < \kappa$. Since π_α is a nice name, its size is bounded by $\lambda \times \lambda \times \omega = \lambda$ (the limit case is easy, by finite support).

Finally, if τ is a relevant $\mathbb{P}_\alpha$-name, let σ be an equivalent nice name. Then $\sigma = \sigma^\alpha_\gamma$ for some γ. Let β be such that $f(\beta) = (\alpha, \gamma)$. □

With this Lemma, the proof of the consistency of $\mathcal{MA}$ is a direct generalisation of the argument for $\mathcal{MAC}$.

Theorem 4.3.18. *Assume that κ is an uncountable regular cardinal such that $2^{<\kappa} = \kappa$ in M, and let $\mathbb{P} = \mathbb{P}_\kappa$ from Lemma 4.3.17. Then for generic $\mathcal{G}$ in $\mathbb{P}$, $M[\mathcal{G}]$ satisfies $\mathcal{MA}$ and $2^\omega = \kappa$.*

We note that by Corollary 2.2.9, the assumption on κ is necessary.

Proof. Since $\mathbb{P}_\kappa$ is ccc, it preserves cofinalities and cardinals. Using the bounds in Lemma 4.3.17 to count nice names, we see that $2^\omega \leq \kappa$ in $M[\mathcal{G}]$. We fix $\lambda < \kappa$ and prove $\mathcal{MA}(\lambda)$. This will also prove $2^\omega \geq \kappa$. We recall (Theorem 2.3.11) that it suffices to prove $\mathcal{MA}(\lambda)$ for posets of size λ.

Let $\mathbb{Q}$ be such a poset, and let D_β be a family of size λ of dense subsets, all in $M[\mathcal{G}]$. Since $\mathbb{P}$ preserves cardinalities, we may assume that the underlying set of $\mathbb{P}$ is λ. Hence $D \subseteq \lambda \times \lambda$, and since $\lambda < \kappa$, we may apply Proposition 4.3.15 to obtain some $\alpha < \kappa$ such that $\mathbb{Q}, D \in M[\mathcal{G}_\alpha]$. Hence, $\mathbb{Q} = \tau_{\mathcal{G}_\alpha}$, where τ is a relevant $\mathbb{P}_\alpha$-name. By the Lemma, $\tau = \pi_\beta$ for some $\beta \geq \alpha$. We therefore have that $M[\mathcal{G}_{\beta+1}] = M[\mathcal{G}_\beta][\mathcal{H}_\beta]$ is an extension by a generic filter $\mathcal{H}_\beta$ in $\mathbb{Q}$. Since $D \in M[\mathcal{G}_\beta]$, we are done. □

End of lecture 33, Nov 16

5. Other stuff

5.1. **The Whitehead problem.** We now discuss an application of the independence results above to a famous question of Whitehead. The result was proved in Shelah [5], but our presentation is based on Eklof [3]. All our groups here are abelian.

Let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian groups; that is, A is a subgroup of B with quotient C. The sequence *splits* if there is homomorphism $s : C \to B$ which is a right inverse of the projection. C is called a *Whitehead group* (or a W-group) if any sequence as above with $A = \mathbb{Z}$ (the integers) splits.

Exercise 5.1.1. (1) *$\mathbb{Z}$ is a Whitehead group*

(2) *Any direct sum of Whitehead groups is a Whitehead group*

It follows that any free abelian group is a Whitehead group. Whitehead asked if the converse is true: is any W-group free? We will show the following.

Theorem 5.1.2. (1) $\diamondsuit$ *implies that any Whitehead group of cardinality* ω_1 *is free*

(2) *There is a non-free group of cardinality* ω_1 *that is Whitehead if* $\mathcal{MA}(\omega_1)$ *holds.*

Hence, by Corollary 3.6.13 and Theorem 4.3.18, both answers are consistent with $\mathcal{ZFC}$ (at least as far as groups of size ω_1 are concerned, but in fact, in general).

We will need the following fact

Fact 5.1.3. *Any subgroup of a free abelian group is free.*

5.1.4. *Homological interpretation.* A map of the sequence $0 \to A \to B_1 \to C \to 0$ to the sequence $0 \to A \to B_2 \to C \to 0$ is a map of groups from B_1 to B_2 that is the identity on A and induces the identity on C. Any such map is an isomorphism. Hence, C is a W-group if and only if any such sequence with $A = \mathbb{Z}$ is isomorphic to the "trivial" sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus C \to C \to 0$. We have the following fact from homological algebra (see, e.g., Eisenbud [2, A3.11]).

Fact 5.1.5. *To any abelian group* A *one may assign another abelian group* $Ext(A, \mathbb{Z})$ *with the following properties.*

(1) *As a set,* $Ext(A, \mathbb{Z})$ *can be identified with isomorphism classes of exact sequences* $0 \to \mathbb{Z} \to B \to A \to 0$ *(the group structure can also be described)*
(2) *Given an exact sequence* $0 \to A \to B \to C \to 0$*, there is an exact sequence*

$$0 \to Hom(C, \mathbb{Z}) \to Hom(B, \mathbb{Z}) \to Hom(A, \mathbb{Z}) \to Ext(C, \mathbb{Z}) \to \\ \to Ext(B, \mathbb{Z}) \to Ext(A, \mathbb{Z}) \to 0 \quad (8)$$

Where $Hom(A, \mathbb{Z})$ *is the group of homomorphisms from* A *to* $\mathbb{Z}$*, and the first three maps are given by composition from the exact sequence.*

Corollary 5.1.6. (1) *Any subgroup of a Whitehead group is Whitehead.*

(2) *Any Whitehead group is torsion free.*
(3) *If (in the sequence above)* B *is a Whitehead group, and any homomorphism* $A \to \mathbb{Z}$ *extends to* B*, then* C *is Whitehead.*
(4) *Suppose* B *is Whitehead but* C *is not. Given an exact sequence* $0 \to \mathbb{Z} \to D \to A \to 0$ *and a splitting* s*, there is a group* E *containing* D*, and a compatible sequence* $0 \to \mathbb{Z} \to E \to B \to 0$*, such that* s *does not extend to a splitting of* $E \to B$*.*

Proof. Exercise (for the last part, may assume $D = \mathbb{Z} \oplus A$, and use the assumption that C is not Whitehead to obtain a homomorphism from A to $\mathbb{Z}$ that does not extend to B. Then modify to obtain an inclusion) □

Proposition 5.1.7. *If* B_i *is a strict chain of groups such that* B_0 *is finitely generated and* B_{i+1}/B_0 *is torsion, then* $B = \cup_i B_i$ *is not Whitehead.*

Proof. Assume that B is Whitehead. Then so is each B_i. Let X be a finite set of generators for B_0. We note that, since each B_i/B_0 is torsion, any map from B_i (or B) to a torsion-free group is determined by its restriction to X.

Let f_i be an enumeration of all maps $f : X \to \mathbb{Z}$. We define a chain of group inclusions C_i and exact sequences $0 \to \mathbb{Z} \to C_i \to B_i \to 0$ as follows. $C_0 = \mathbb{Z} \oplus B_0$. Given $C_i \to B_i$, let s be a splitting, which restrict to $x \mapsto (f_i, x)$ on X, if possible. Then C_{i+1} is as in the last part of Corollary 5.1.6.

Let $C = \cup_i C_i$. Then, as a set, $C = \mathbb{Z} \times B$, and we have an exact sequence $0 \to \mathbb{Z} \to C \to B \to 0$. If s is a splitting, the first coordinate of the restriction of s to X is some f_i. Then s restricted to B_i is the splitting we chose on step i, but it extends to B_{i+1}, contradicting the choice of C_{i+1}. Hence B has a non-split extension, contradicting Whiteheadness. □

5.1.8. *Pure subgroups.* From now on, we only consider torsion-free groups. Contrary to popular belief, the union of a chain of free groups need not be free (consider $\mathbb{Z}[\frac{1}{2}] = \bigcup_n \langle \frac{1}{2^n} \rangle$). However, the problem only results from impurity: Recall that a *pure subgroup* of a torsion free group A is a subgroup B closed under division by integers. In other words, A/B is torsion free. Clearly, if B is pure in A, and C is an intermediate group, then B is pure in C. Conversely, if B is pure in C and C is pure in A, then B is pure in A.

Any subgroup B is contained in a smallest pure subgroup called the *pure closure* of B, that has the same rank as B (the *rank* of B is the $\mathbb{Q}$-dimension of $\mathbb{Q} \otimes B$). By the above fact, a subgroup of rank r of a free group is generated by r elements, so in particular, the pure closure of every finitely generated subgroup of a free group is finitely generated. In the countable case, there is the following converse.

End of lecture 34, Nov 18

Proposition 5.1.9. *If A is countable and torsion free, and any finitely generated subgroup is contained in a finitely generated pure subgroup, then A is free.*

Proof. By assumption, we may write A as a countable union of finitely generated pure subgroups A_i. As each quotient is torsion-free and finitely generated, it is free. In particular, it is Whitehead, so A_i is a direct summand of A_{i+1}. Hence we may choose a compatible system of bases B_i for the A_i. The union $\cup_i B_i$ is a basis for A. □

Combining Propositions 5.1.7 and 5.1.9, we may give a positive answer to Whitehead's problem in the countable case:

Theorem 5.1.10. *Any countable Whitehead group is free*

Proof. Let A be a countable Whitehead group, let B_0 be a finitely generated subgroup, and let B be the pure closure of B_0. If B is not finitely generated, it can be written as a union of a strictly increasing chain B_i. Since B is the pure closure, each B_i/B_0 is torsion, contradicting 5.1.7. □

We now turn to discuss the uncountable situation. First, from the countable case, we have the following.

Corollary 5.1.11. *Any countable subgroup of a Whitehead group is free*

Call a group *κ-free* if any subgroup of cardinality less than κ is free. Hence, a group is torsion-free precisely if it is ω-free, and any Whitehead group is ω_1-free. We may thus generalise purity as follows: A subgroup $B \subset A$ is *κ-pure* if A/B is κ-free. We shall study groups satisfying a condition analogous to that of Proposition 5.1.9.

Definition 5.1.12. Let A be an ω_1-free group of cardinality ω_1. A satisfies the *Chase condition* if any countable subgroup is contained in a countable ω_1-pure subgroup.

We say that a chain of sets (or groups) A_α is *smooth* if $A_\alpha = \cup_{\beta<\alpha} A_{\beta+1}$ for all α. As in the proof of 5.1.9.

Exercise 5.1.13. *A satisfies the Chase condition if and only if it is the union of a smooth chain A_α of countable free groups of length ω_1, such that for each successor ordinal α, A_α is ω_1-pure in A.*

5.1.14. We note that being ω_1-pure is not closed under unions of chains. Hence it makes sense to ask what happens on limits stages. Let E be the set of countable limit ordinals α for which A_α is not ω_1-pure. (so strictly speaking, E depends on the choice of the sequence, but not seriously, as seen below).

Theorem 5.1.15. *Assume A satisfies the Chase condition. Then it is free if and only if E is not stationary.*

Proof. Assume E is not stationary, and let C be a club that does not intersect E. Then $\cup_{\alpha\in C} A_\alpha = A$, and this is a smooth chain, and each A_α for $\alpha \in C$ is ω_1-pure in A. Hence, each successive quotient is free (since it is countable), so the union is free.

Assume now that A is free, and let X be a basis. By the usual method, we may find a smooth sequence $X_\alpha \subseteq X$ such that $\langle X_\alpha \rangle$ is a cofinal subset of the A_α (exercise). The set of β such that $A_\beta = \langle X_\alpha \rangle$ for some α is a club. Such a β cannot be in E, since A/A_β is freely generated by the images of $X \setminus X_\alpha$. Therefore E is not stationary. □

End of lecture 35, Nov 21

The last Theorem is an analogue of Proposition 5.1.9. To produce an analogue of Theorem 5.1.10, we need the following analogue of Proposition 5.1.7. We continue using the terminology of 5.1.14.

Theorem 5.1.16. *Assume $\diamondsuit$. If E is stationary, then A is not Whitehead.*

Proof. We use a version of $\diamondsuit$ that allows us to guess a function $f : A_\alpha \to \mathbb{Z}$ (as in Proposition 1.4.19). Let f_α be such a diamond sequence. Now the proof is exactly as in Proposition 5.1.7: we construct a sequence of exact sequences $0 \to \mathbb{Z} \to C_\alpha \to A_\alpha \to 0$, where the C_α is a smooth chain, and at stage $\alpha + 1$, with $\alpha \in E$, we destroy the chance of f_α to extend to a splitting of the union (this is possible since when $\alpha \in E$, $A_{\alpha+1}/A_\alpha$ is not free, hence not Whitehead, by Theorem 5.1.10).

Let $C = \cup_{\alpha<\omega_1} C_\alpha$. If the exact sequence $0 \to \mathbb{Z} \to C \to A \to 0$ splits, with splitting map corresponding to $f : A \to \mathbb{Z}$, then, since E is stationary, the restriction of f to A_α is f_α for some $\alpha \in E$. This is a contradiction. □

Corollary 5.1.17. *Assume $\diamondsuit$. Then any Whitehead group of cardinality ω_1 satisfies the Chase condition.*

Proof. Assume B violates the claim. Then there is a countable subgroup A_0 that is not contained in any countable ω_1-pure subgroup. Hence, for any countable C containing A_0, there is a countable C_1 containing C such that C_1/C is not free. Use this to build a smooth chain A_α, $\alpha < \omega_1$ of countable subgroups with $A_{\alpha+1}/A_\alpha$ not free. The union is not Whitehead by Theorem 5.1.16, contradicting Corollary 5.1.6. □

We may now give a positive answer to Whitehead's problem for ω_1, if we assume $\diamondsuit$.

Proof of 5.1.2.(1). Let A be a Whitehead group of cardinality ω_1. By Corollary 5.1.17, A satisfies the Chase condition. Hence the exceptional set E is not stationary by Theorem 5.1.16, so A is free by Theorem 5.1.15. □

We now turn to the other direction. We first make the following observation.

Proposition 5.1.18. *There is a group of cardinality ω_1 which satisfies the Chase condition, but which is not free.*

Proof. It is enough to construct a smooth chain A_α of countable free groups, such that A_β/A_α is free precisely if α is not a limit ordinal.

When $\beta = \alpha + 1$ and α is not limit, we let $A_\beta = A_\alpha \oplus \mathbb{Z}$. When β is limit, $A_\beta = \cup_{\alpha<\beta} A_\alpha$. In both cases, it is clear that the condition is satisfied.

Now assume $\beta = \alpha + 1$, where α is limit. We may write $A_\alpha = \cup_{i\in\omega} B_i$, where $B_i = A_{\alpha_i+1}$ for some sequence $\alpha_i < \omega_1$. Since each B_i and each successive quotient is free, we may choose a compatible system of bases X_i for B_i. For each $i > 1$, choose an element $x_i \in X_i \setminus X_{i-1}$, and let $Y_i = X_i \setminus \{x_i\}$, $Y = \cup Y_i$. Let $P = \prod_i \langle x_i \rangle$. We view the subgroup of A_α generated by the x_i as a subgroup of P. In particular, A_α may by viewed as as subgroup of $\langle Y \rangle \oplus P$. We let $A_{\alpha+1}$ be the subgroup of this last group generated by A_α and the elements z_n of P given by $z_n = \sum_{i\geq n} \frac{i!}{n!} x_i$.

We need to verify the conditions. We note the relation $z_n - (n+1)z_{n+1} = x_n$. This implies that $A_{\alpha+1}$ is generated by Y and the z_i, and clearly there are no relations. Hence $A_{\alpha+1}$ is free. More generally, $A_{\alpha+1}/B_i$ is freely generated by the part of the basis that does not involve X_i. This is sufficient to show that $A_{\alpha+1}/A_\beta$ is free when β is any successor. Finally, any finite truncation of z_1 is in A_α, and the n-th such truncation is given by $z_1 - n!z_n$. So in $A_{\alpha+1}/A_\alpha$, z_1 is a non-zero element divisible by any integer. Hence this group cannot be free. □

End of lecture 36, Nov 28

Theorem 5.1.2.(2) will thus follow once we prove:

Theorem 5.1.19. *Assume $\mathcal{MA}(\omega_1)$. Then any group of size ω_1 satisfying the Chase condition is Whitehead.*

Let A be a group of size ω_1, and let $0 \to C \to B \to A \to 0$ be an exact sequence, where C is countable. As before, we mays assume B to be $A \times C$ as a set, and that the map $B \to A$ is the projection. A section $s : A \to A \times C$ is then determined uniquely by its second component $\phi_s : A \to C$.

We will apply Martin's axiom with the following order $\mathbb{P}$. An element of $\mathbb{P}$ is a pair (f, S), where $S \subseteq A$ is finite and generates a pure subgroup A' of A, and $f : A' \to C$ is a function that correspond to a section of $B \to A$ restricted to A'. $\mathbb{P}$ is ordered by: $(f, S) \leq (g, T)$ if $f\restriction_S$ extends $g\restriction_T$.

Proposition 5.1.20. *Assume A is ω_1-free. Then for any $a \in A$, the set $D_a = \{(f, S)|a \in S\}$ is dense in $\mathbb{P}$.*

Proof. Let (g, T) be an arbitrary element of $\mathbb{P}$. The group $\langle T, a\rangle$ is countable, hence so is its pure closure (since A is torsion-free). Hence the pure closure is free, and therefore finitely generated, by a finite set S, which we may assume to contain T and a. Since $\langle T \rangle$ is pure, $\langle S\rangle/\langle T\rangle$ is free. Hence we may extend the section corresponding to g to a section on $\langle S\rangle$. □

The use of the Chase condition is through the following Lemma.

Lemma 5.1.21. *Assume, with notation as above, that A satisfies the Chase condition. Then for any uncountable antichain X of $\mathbb{P}$ there is a free pure subgroup A' of A and an uncountable $Y \subseteq X$, such that $S \subseteq A'$ for all $(f, S) \in Y$.*

Proposition 5.1.22. *Assume that A satisfies the Chase condition. Then $\mathbb{P}$ is ccc.*

Proof. Assume X is an uncountable anti-chain. By Lemma 5.1.21, we may assume that A is free (note that a pure subgroup of a pure subgroup is pure in the larger group). Let U be a basis for A. By Proposition 5.1.20, we may assume that for each $(f, S) \in X$, $S \subseteq U$. But now (f, S) and (g, T) are compatible precisely if f and g agree on $S \cap T$. Hence $\{f\restriction_S | (f, S) \in X\}$ determines an uncountable anti-chain in $\mathbb{F}n(U, C)$, a contradiction (since C is countable). □

Proof of Theorem 5.1.19. Let A be a group satisfying the chase condition, $0 \to \mathbb{Z} \to B \to A \to 0$ and exact sequence, $\mathbb{P}$ as above. Let $\mathcal{F}$ be a generic filter for the dense sets D_a as in Proposition 5.1.20, for all $a \in A$. Then $\cup\mathcal{F}$ determines a splitting. □

Proof of Lemma 5.1.21. Using the Δ-system lemma, we may assume: there is a pure subgroup T, such that $\langle S_1\rangle \cap \langle S_2\rangle = T$ for any distinct elements (f_1, S_1) and (f_2, S_2) of X. Let (f_α, S_α) be an enumeration of X, $G_\alpha = \langle S_\alpha\rangle$. We construct a smooth chain A_α, with $A_0 = T$, and a strictly increasing function $u : \omega_1 \to \omega_1$, such that $G_{u(\alpha)} \subseteq A_{\alpha+1}$.

Assuming A_α was defined, let C be a countable ω_1-pure subgroup of A containing A_α. Any element of $C \setminus T$ belongs to at most one G_α, hence, since C is countable, there is some $u(\alpha)$ such that $G_{u(\alpha)} \cap C = T$. Let $A_{\alpha+1}$ be the pure closure of $A_\alpha + G_{u(\alpha)}$. We note that $A_{\alpha+1} \cap C = A_\alpha$, so $A_{\alpha+1}/A_\alpha = A_{\alpha+1}/(A_{\alpha+1} \cap C) \subseteq A/C$. The last group is ω_1-free by the choice of C, so $A_{\alpha+1}/A_\alpha$ is free.

Now let $A' = \cup_\alpha A_\alpha$. Since each A_α is pure, so is A'. Since each $A_{\alpha+1}/A_\alpha$ is free, so is A'. Take Y to be the elements of X indexed by the image of u. □

Proof of Theorem 5.1.2.(2). Let A be the group from Proposition 5.1.18. □

We note that it follows that there is such a group of arbitrary cardinality, since an arbitrary direct sum of Whitehead groups is Whitehead. End of lecture 37, Nov 30

5.2. **More on absoluteness.** We would like to understand to which extent forcing affects everyday mathematics. Recall that if a model N of $\mathcal{ZFC}$ is obtained from M by forcing, then $M \subseteq N$ is transitive, and M and N agree on the ordinals. Hence we may apply basic absoluteness results.

Example 5.2.1. *Elementary number theory statements use quantification over natural numbers and then arithmetic operations (e.g., $\forall x, y, z, n > 0 \in \omega(x^{n+2}+y^{n+2} \neq z^{n+2})$). Since ω is absolute, these are Δ_0 statements over an absolute set. The arithmetic operations are not originally in the language, but they are defined by recursion using absolute (even Δ_0) formulas. Hence all such statements are absolute. We call a statement of this form* arithmetic.

Then there is calculus. The real numbers are coded in some way as sets of integers or rationals. Since the set of real numbers is then in bijection with $\mathcal{P}\omega$, we cannot expect it to be absolute. However, many statements in calculus use only few quantifiers.

Example 5.2.2. *The statement "f is continuous at x" for a (given) real valued function f appears to have three alterations of quantifiers over real numbers. However, the ϵ and δ of the standard definition can be taken rational, so in fact, it involves only one quantifier over the reals.*

We define the quantifier complexity of a formula as follows.

Definition 5.2.3. The *analytic hierarchy* is defined as follows. We set $\Pi_0 = \Delta_0^\omega$, the set of arithmetic formulas. For each n, $\Sigma_n = \{\neg\phi | \phi \in \Pi_n\}$, and $\Pi_{n+1} = \{\forall x \in 2^\omega \phi | \phi \in \Sigma_n\}$, all up to equivalence (with respect to $\mathcal{ZFC}$).

Hence, the formula in the example is in Π_1.

Remark 5.2.4. Standard coding results show that a sequence of universal quantifiers can be replaced by one universal quantifier. Also, with more coding results, it is possible to move any quantifier over ω to the right of any quantifier over 2^ω. Hence, to determine that $\phi \in \Pi_n$, it is enough to count the number of alterations of quantifiers over sets in any prenex normal form, disregarding quantifiers over ω.

We will use capital letters for variables ranging over sets or functions of integers.

We remark that the set of real functions is not covered by any analytic formula. However, interesting sets of functions, such as continuous, differentiable, measurable, etc., are equinumerous with the reals.

Next, we turn to algebra and logic. Elementary algebraic statements (e.g., the characteristic of a field is 0 or prime) are provable in the corresponding theory. Since a proof can be coded within ω, each such statement is again arithmetic. Of course, we have seen above that non-elementary statements may depend on set theory.

Most examples so far are rather artificial in that it is difficult to come up with an application of a set theoretic axiom (say, the $\mathcal{GCH}$) in their proof. In model theory, one common method is to use saturated models, whose existence does depend on set theoretic assumptions. However, in many cases, the resulting statement is independent.

Example 5.2.5. *There is a criterion for quantifier elimination using saturated models. However, the statement of quantifier elimination in a particular theory says that any formula is equivalent to a formula of a given type. This is again an arithmetic statement, so does not depend on the existence of saturated models.*

The following two examples are taken from Baldwin [1, § 5].

Example 5.2.6. *The statement that that theory is stable or NIP can be phrased by saying that no formula in the language codes a subsets of a certain type in an arbitrary large finite set. For example, $\phi(x, y)$ is unstable if there are arbitrarily long finite sequences x_i and y_j such that $\phi(x_i, y_j)$ if and only if $i < j$. This is again a collection of arithmetic conditions, so being stable is arithmetic.*

The following is again a non-arithmetic example, but in Π_1.

Example 5.2.7. *A theory T is not ω-stable if and only if there is a consistent set of formulas $\phi_i^{s(i)}(x_s, a_{s_i})$, where $s \in 2^\omega$, $i \in \omega$ and $\phi^1 = \neg\phi$.*

As mentioned above, our absoluteness results so far account for the arithmetic statements, but not for the more general statements as in the last example. The following theorem from Shoenfield [6] takes care of these (and more).

Theorem 5.2.8 (Shoenfield absoluteness)**.** *Any Σ_2 statement is absolute for transitive inclusions $M \subseteq N$ that preserve the ordinals.*

Obviously, it follows that any Π_2 statement is absolute as well. Also, it follows that any Π_3 statement is preserved downwards, and any Σ_3 statement is preserved upwards. We note that this Theorem holds in $\mathcal{ZF}$ (and so can be used to eliminate the Axiom of Choice).

Sketch of proof. The main point is that for inclusions as in the Theorem, the statement "x well-orders y", which we abbreviate $x\mathcal{W}y$, is absolute. Indeed, the standard definition is universal (in subsets of y), but it can also be expressed by: There is a function $F : y \to \alpha$, with α an ordinal, such that $(a, b) \in x$ if and only if $F(a) \le F(b)$. (namely, the rank with respect to x is such a function. Note that this is in $\mathcal{ZF}$.) Since, by assumption, the ordinals are preserved, this is existential over an absolute formula.

The rest of the proof uses coding. It turns out that any Π_1 statement $\phi(x)$, where x consists of variables ranging over ω or subsets of ω, is equivalent to a statement $g(x)\mathcal{W}\omega$, where g is arithmetic. We omit the proof, but there is a proof of a similar statement below. It follows by the above that any Π_1 statement is absolute.

We thus have that a Σ_2 formula $\exists T\forall S\phi(S, T, x)$ can be written as $\exists T(g(x)\mathcal{W}\omega)$ for arithmetic g, which, by the first remark, can be written as

$$\exists T\exists\alpha\exists F : \omega \to \alpha(\forall i, j \in \omega((i, j) \in g(x) \iff F(i) \le F(j))) \tag{9}$$

with α ranging over ordinals. By exchanging quantifiers and coding, we may write this as

$$\exists\alpha\exists F : \omega \to \alpha \times \omega(\forall i(\psi(F\restriction_i, x))) \tag{10}$$ {eq:abs}

with ψ arithmetic. Since α ranges over ordinals, which are preserved by assumption, the result of the Theorem follows once we show the following claim.

Claim 5.2.9. *For any arithmetic $\psi(z, x)$, there are arithmetic functions $g(\alpha, x)$ and $h(\alpha, x)$ such that*

$$\exists F : \omega \to \alpha \times \omega(\forall i(\psi(F\restriction_i, x))) \tag{11}$$

is equivalent to $\neg(g(\alpha, x)\mathcal{W}h(\alpha, x))$.

Proof. We first define h. $h(\alpha, x)$ is the set of functions $u : n \to \alpha \times \omega$ with the property that $\psi(f\restriction_i, x)$ is true for all $i < n$. Note that h is arithmetic. $g(\alpha, x)$ orders $h(\alpha, x)$ lexicographically: Given two such functions u and v, $u \le v$ if u is an extension of v, or $u(i) < v(i)$ on the first i where they are both defined and disagree. Again, this is arithmetic.

Assume there is an infinite descending chain. Since $\alpha\times\omega$ is well-ordered, we may find an infinite sub-sequence of extensions. The union gives a function $F : \omega \to \alpha\times\omega$ as required. On the other hand, such a function restricts to an infinite descending chain. ☐

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End of lecture 38, Dec 2

5.3. **Independence of $\mathcal{AC}$.** In this section we start with a model of $\mathcal{ZFC}$, and construct a model of $\mathcal{ZF}$ that violates $\mathcal{AC}$. Since a classical result of Gödel shows that $\mathcal{AC}$ is consistent with $\mathcal{ZF}$, this will establish independence. The material in this section is based on Smullyan and Fitting [7, § 20].

By Proposition 3.3.10, we cannot construct a model violating $\mathcal{AC}$ using usual forcing. Instead, we develop an equivariant version, where we are given a group action. We repeat the constructions and statements using elements fixed under the action. We will then show that the truth of universal formulas is preserved when passing between the equivariant and the usual version.

The situation is a bit similar to the following. Let $f : X = \mathbb{C}^* \to Y = \mathbb{C}^*$ (punctured complex plane) be the function $f(z) = z^2$. We may think of f as defining a family of two element sets ("socks"), parametrised by Y. Assume that we would like to find an *analytic* choice function, i.e., a function $s : Y \to X$ such that $f(s(z)) = z$ (in other words, we would like to analytically choose a square root). If s is such a function, it determines a function $s \circ f : X \to X$, with the property that it maps each fibre into itself, and is constant on the fibre (indeed, it attaches to each sock the chosen one from its pair). Conversely, any such function comes from a section$s$. Hence, the problem of finding a choice function on Y can be translated to the problem of finding a "choice function" on X, which is fixed by the action of the automorphisms of X over Y. Despite the fact that there are functions on X that preserve the fibres (e.g., the identity), none of them is invariant, so Choice for invariant functions fails.

5.3.1. Let $\mathbb{P}$ be a poset in a ctm M, and let u be an automorphism of $\mathbb{P}$. Then u extends to a map on $\mathbb{P}$-names (again denoted by u), and on formulas over the language of names.

Exercise 5.3.2. *If u is an automorphism of $\mathbb{P}$ in M, then for any formula ϕ (in the language of names) and $p \in \mathbb{P}$, $p \Vdash \phi$ if and only if $u(p) \Vdash u(\phi)$ (See also Kunen [4, § VII.7.13], but the proof is easier in this case)*

Definition 5.3.3. Let $\mathbb{P}$ be a poset in a ctm M, and let $G = (G_0 \geq G_1 \geq \dots)$ be a descending sequence of groups of automorphisms of $\mathbb{P}$. A name σ is *strongly G-invariant* if it is invariant under some G_i, and for each $(\theta, p) \in \sigma$, θ is strongly G-invariant. We denote by $M_G^{\mathbb{P}}$ the set of strongly G-invariant names.

Exercise 5.3.4. *For any element m of M, $\check{m}$ is strongly G-invariant.*

We note that τ (the name for the generic filter) is not, in general, invariant.

Definition 5.3.5. Let $\mathcal{F}$ be a generic filter in $\mathbb{P} \in M$. We let $M[\mathcal{F}]^G$ be the set of elements that have a strongly invariant name.

Lemma 5.3.6. *Let σ be a strongly invariant name, and assume that $x \subseteq \sigma_{\mathcal{F}}$, with $x \in M[\mathcal{F}]^G$. Then there is a strongly invariant θ with $\mathrm{dom}(\theta) \subseteq \mathrm{dom}(\sigma)$, such that $\theta_{\mathcal{F}} = x$.*

Proof. Let θ_0 be any strongly invariant name for x, and let $\theta = \{(\pi, p) | \pi \in \mathrm{dom}(\sigma), p \Vdash \pi \in \theta_0\}$. Then θ is a name for x with $\mathrm{dom}(\theta) \subseteq \mathrm{dom}(\sigma)$. Let G_i be a group in the sequence fixing both σ and θ_0. If $g \in G_i$ and $(\pi, p) \in \theta$, then $g\pi \in \mathrm{dom}(\sigma)$ since $g\sigma = \sigma$, and $gp \Vdash g\pi \in g\theta_0 = \theta_0$ by 5.3.2. Hence $(g\pi, gp) \in \theta$. □

Proposition 5.3.7. *$M[\mathcal{F}]^G$ is a transitive substructure of $M[\mathcal{F}]$. Hence all Δ_0 formulas are absolute for $M[\mathcal{F}]^G \subseteq M[\mathcal{F}]$.*

Proof. Exercise □

We will want to know that strongly invariant names are preserved under the action of G_0. This requires the following condition.

Definition 5.3.8. We say that the sequence G is *normal* if any $g \in G_0$ normalises some G_i ($g^{-1}G_i g = G_i$).

Proposition 5.3.9. *Assume that G is normal, and $g \in G_0$. If σ is strongly G-invariant, then so is $g(\sigma)$.*

Proof. Exercise □

Definition 5.3.10. Let ϕ be a formula with all names strongly invariant. We say that $p \in \mathbb{P}$ *G-forces* ϕ ($p \Vdash_G \phi$) if $\phi_{\mathcal{F}}$ holds in $M[\mathcal{F}]^G$ whenever $p \in \mathcal{F}$.

Exercise 5.3.11. *Assume that G is normal. Then for any $g \in G$, $p \Vdash_G \phi$ if and only if $gp \Vdash_G g\phi$.*

We say that G is in M if it is in M as sequence of groups, and the action of G_0 on $\mathbb{P}$ is also in M. We note that in this case, the set of strongly invariant names is definable.

We now repeat the basic facts on forcing in this equivariant setting. We fix a normal sequence G in M.

Theorem 5.3.12. *Assume that G is normal and in M. The relation $p \Vdash_G \phi$ is definable. If a strongly invariant ϕ holds in $M[\mathcal{F}]^G$, then $p \Vdash_G \phi$ for some $p \in \mathcal{F}$.*

Proof. Exercise (same as Proposition 3.4.2, but use just strongly invariant names for the existential quantifier. Note that for quantifier free formulas ϕ, $\Vdash$ and $\Vdash_G$ coincide) □

Theorem 5.3.13. *Let M be a ctm, $\mathbb{P}$ a poset, and $\mathcal{F}$ a generic filter. For any normal sequence G in M, $M[\mathcal{F}]^G$ is a model of $\mathcal{ZF}$*

Proof. Foundation and Infinity are obvious. Extensionality holds in $M[\mathcal{F}]$ and is given by a universal formula, so is true also in $M[\mathcal{F}]^G$. For the Pair and Union axioms, the names given in the proof of Proposition 3.3.10 are strongly invariant (if the data is strongly invariant), so the same proof works.

So we need to verify Power set, Comprehension and Replacement. For Power set, let $x \in M[\mathcal{F}]^G$ have strongly invariant name σ. Let S be the set of strongly invariant elements of $\mathcal{P}(\mathrm{dom}(\sigma) \times \mathbb{P})$. Since G is in M, S is in M as well. By Lemma 5.3.6, $S \times \{\mathbf{1}\}$ is a name for a set containing the power set of x.

For comprehension, as in the proof of Theorem 3.4.6, let σ by a strongly invariant name, and let $\phi(x, y)$ be a formula, in which all constants are strongly invariant. Let

$$\theta = \{(\pi, p) | \pi \in \mathrm{dom}(\sigma), p \Vdash_G \pi \in \sigma \wedge \phi(\pi, \sigma)\} \tag{12}$$

This is a name by Theorem 5.3.12, and as before, it is clear that θ_G is the required element. It is strongly invariant by Exercise 5.3.11. Again, the proof for Replacement is similar. □

Exercise 5.3.14. *Where does the proof of $\mathcal{AC}$ in the regular case (Proposition 3.3.10) fails in the equivariant case?*

End of lecture 39, Dec 5

We are now in position to produce a model that violates $\mathcal{AC}$. Let M be a ctm, and let $\mathbb{P} = \mathbb{F}n(\omega \times \omega, 2)$. As in Proposition 3.5.1, forcing with this poset produces a model $N = M[\mathcal{F}]$ with ω new subsets of ω. Names for these new sets can be given explicitly by

$$\sigma_i = \{(\check{n}, p) | n \in \omega, p(i, n) = 1\} \tag{13}$$

Let $\omega = \cup_i B_i$ be a partition of ω in M to infinitely many infinite classes, such that $k \geq i$ for all $k \in B_i$, and let

$$\tau_n = \{(\sigma_i, \mathbf{1}) | i \in B_n\} \tag{14}$$

We will show that $\theta = \{(\tau_i, \mathbf{1})\}$ is strongly invariant for a suitable G, and that $\theta_{\mathcal{F}} = \{\tau_{i\mathcal{F}}\}$ has no choice function in $M[\mathcal{F}]^G$.

To define an action on $\mathbb{P}$, we note that it is enough to define an action on ω (acting on the first coordinate of $\omega \times \omega$, and therefore on $\mathbb{P}$). We call permutation $u : \omega \to \omega$ *special* if it has finite support, and preserves each B_i (The *support* of a permutation $u : A \to A$ is the set of $a \in A$ with $u(a) \neq a$). We let G_n be the group of special permutations with support disjoint from n. Clearly, the G_i are in M.

Exercise 5.3.15. *The sequence $G = (G_i)$ is normal*

Hence $M[\mathcal{F}]^G$ is a model of $\mathcal{ZF}$.

Proposition 5.3.16. *T is strongly invariant*

Proof. We compute the action of G_0 on the relevant names. Let $g \in G_0$. Then

$$g\sigma_i = \{(g\check{n}, gp) | p(i, n) = 1\} = \{(\check{n}, p) | p(gi, n) = 1\} = \sigma_{g(i)} \tag{15}$$

$$g\tau_n = \{(g\sigma_i, \mathbf{1}) | \sigma_i \in B_n\} = \{(\sigma_i, \mathbf{1}) | \sigma_{g^{-1}(i)} \in B_n\} = \tau_n \tag{16}$$

(since g preserves B_n)

$$g\theta = \{(g\tau_n, \mathbf{1})\} = \theta \tag{17}$$

It follows that σ_i is invariant under all elements of G_{i+1}. The rest is obvious. □

Theorem 5.3.17. *With the above notation, $M[\mathcal{F}]^G$ is a model of $\mathcal{ZF}$ in which $\mathcal{AC}$ fails.*

Proof. Let $T = \theta_{\mathcal{F}}$, $t_n = \tau_{n\mathcal{F}}$ and $s_n = \sigma_{n\mathcal{F}}$. Hence, the s_i are distinct subsets of ω not in M, $t_n = \{s_i | i \in B_n\}$ and $T = \{t_i\}$, and all are in $M[\mathcal{F}]^G$. We prove that there is no choice function for T.

Assume that such a function c exists, and let π be a strongly invariant name for it. Let n be such that G_n fixes π, and let $p \in \mathcal{F}$ be such that p G-forces that π is a function, and determines its value on t_n, $c(t_n) = s_k$.

Let $q_k(i) = q(k, i)$, and consider $D_k = \{q \in \mathbb{P} | \exists l > k, l \in B_n, q_l = q_k\}$. Then, since B_n is infinite, D_k is dense below p, so there is some q in $\mathcal{F}$ extending p, such that $q_k = q_l$. Let g be the transposition (kl). Then g is special since $k, l \in B_n$, $g \in G_n$ since $k, l \geq n$, and $gq = q$. By Exercise 5.3.11, $gq \Vdash_G g\pi(g\tau_n) = g\sigma_k$, hence $gq = q \Vdash_G \pi(t_n) = \sigma_{g(k)} = \sigma_l$, since π is fixed by G_n (and τ_n is fixed by G_0). Since $q \in \mathcal{F}$, this is a contradiction. □

Remark 5.3.18. Note that the poset we used preserves cardinals, and being a cardinal is expressible by a universal formula, so all cardinals from the original model M are preserved.

End of lecture 40, Dec 7

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Index